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Journal of Approximation Theory 122 (2003) 160–207

Journal of Approximation Theory

http://www.elsevier.com/locate/jat

The uniform convergence of subsequences of the last intermediate row of the Padé table ☆

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Received 29 April 2001; accepted in revised form 10 April 2003

Communicated by Guillermo López Lagomasino

Abstract

In the work the uniform convergence of rows of the Padé approximants for a meromorphic function $a(z)$ is studied. The complete description of the asymptotic behavior of denominators $Q_n(z)$ of the Padé approximants is obtained for the $(\lambda - 1)$th row. Here λ is the number of the poles of $a(z)$. The limits of all convergent subsequences of $\{Q_n(z)\}$ are explicitly computed. These limits form a family of polynomials which is parametrized by a monothetic subgroup $\mathbb{F}$ of the torus $\mathbb{T}^\nu$. The group $\mathbb{F}$ is constructed via the arguments $\Theta_1, \ldots, \Theta_\nu$ of those poles of $a(z)$ of the maximal modulus that have the maximal multiplicity.

1. Introduction

The convergence of row sequences of the Padé table for meromorphic functions is a subject of numerous works (see, e.g., [11,12,14,19,20,22,25,29–32]). The first result obtained by de Montessus de Ballore in 1902 [25]. In this work de Montessus showed that the row sequence of the Padé approximants $\pi_{n,\lambda}(z)$ of type (n, λ) for a meromorphic function $a(z)$ converges to $a(z)$ as $n \to \infty$ in a disk $|z| < R$ with the exception of neighborhoods of the poles of $a(z)$. Here λ is the number of the poles (counting multiplicity) of $a(z)$ in the disk. The key to the problem was the fact,

☆ This work was supported by Russian Foundation for Basic Research under Grant 01-01-96422.
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doi:10.1016/S0021-9045(03)00079-0

discovered by Hadamard [20], that the zeros of the denominators $Q_{n,\lambda}(z)$ of $\pi_{n,\lambda}(z)$ tend to the poles of $a(z)$.

Lin [22] and Sidi [29] obtained a sufficient condition for the convergence of some intermediate rows of the Padé table. To describe their results, we introduce some needed notation. Let $a(z)$ be a function which is meromorphic in the disk $D_R = \{z \in \mathbb{C} \mid |z| < R\}$ and analytic at the origin. Let $z_1, \dots, z_\ell$ be its distinct poles of multiplicities $s_1, \dots, s_\ell$, respectively, and let $\lambda = s_1 + \cdots + s_\ell$ be the number of its poles in the disk D_R. Suppose that

$$\rho \equiv |z_1| = \cdots = |z_\mu| > |z_{\mu+1}| \geqslant \cdots \geqslant |z_\ell|.$$

The poles $z_1, \dots, z_\mu$ of the maximal modulus are ordered in such a way that $s_1 \geqslant \cdots \geqslant s_\mu$. If

$$m = \sum_{j=1}^{\ell} s_j \quad \left(m = \sum_{j=\mu+1}^{\ell} s_j \right),$$

then, by de Montessus's theorem, $\pi_{n,m}(z)$ converges to $a(z)$, uniformly on any compact subset of the set $D_R \backslash \{z_1, \dots, z_\ell\}$ $(D_\rho \backslash \{z_{\mu+1}, \dots, z_\ell\})$. If

$$\sum_{j=\mu+1}^{\ell} s_j < m < \sum_{j=1}^{\ell} s_j,$$

then mth row of the Padé table is said to be an *intermediate row*.

In [22,29] it is shown that $\lim_{n \to \infty} \pi_{n,m}(z)$ exists for a mth intermediate row if there is a unique solution to the following integer programming problem:

$$\text{maximize} \sum_{j=1}^{\mu} (s_j \sigma_j - \sigma_j^2) \quad \text{over}(\sigma_1, \dots, \sigma_\mu) \tag{1.1}$$

subject to $\sum_{j=1}^{\mu} \sigma_j = m - \sum_{j=\mu+1}^{\ell} s_j$ and $0 \leqslant \sigma_j \leqslant s_j$, σ_j are integers. In particular, it is easily seen that for $m = \sum_{j=1}^{\ell} s_j - 1$ (for the last intermediate row) a unique solution of problem (1.1) exists iff $s_1 > s_2$. Hence if the function $a(z)$ has a single pole of the maximal multiplicity among the poles of the maximal modulus, then there exists $\lim_{n \to \infty} \pi_{n,\lambda-1}(z)$.

In the work of Liu and Saff [24] the convergence of intermediate rows of the Walsh array of the best rational approximations to a meromorphic function was investigated (in the special case when $a(z)$ has poles at just two point at the boundary see [28]). The authors obtained also a uniqueness criterion and explicit formulas for a unique solution of problem (1.1). The convergence of intermediate rows for a multipoint Padé approximations was studied in [23].

For intermediate rows not satisfying the uniqueness criterion ("bad" intermediate rows, by terminology of Liu and Saff) Liu [23, Section 5] gave an example to explain the complexity of convergence. He investigated an asymptotic behavior of poles of $\pi_{n,1}(z)$ for a function with exactly two different simple poles lying on the unite circle. It turns out that the set of the limit points of the zeros of $Q_{n,1}(z)$ could make up an arc of a circle in $|z| < 1$.

In the general case, the complete description of the limit point set of the poles of $\pi_{n,m}$ as $n \to \infty$, does not exist for any row. However, there is an intermediate row for which this description can be obtained. It turns out that for the last intermediate row $(m = \lambda - 1)$ the complete theory of the uniform convergence of the sequence $\pi_{n,\lambda-1}(z)$ can be constructed. This means that we can explicitly calculate the limits of all convergent subsequences of the sequence of the denominators $Q_{n,\lambda-1}(z)$ as $n \to \infty$. Hence for this row we can explicitly describe the set of limit points of the poles of $\pi_{n,\lambda-1}(z)$. In particular, we will show that the inequality $s_1 > s_2$ is the necessary convergence condition for the last intermediate row. Moreover, the example of Liu will be significantly generalized.

In the present work we will establish that the asymptotic behavior of the denominators $Q_{n,\lambda-1}(z)$ is defined by an arithmetic nature of these poles of $a(z)$ that have the maximal modulus and the maximal multiplicity. Our approach to the problem can briefly be described as follows.

Let us represent the meromorphic function $a(z)$ in the form:

$$a(z) = b(z) + r(z).$$

Here the function $b(z)$ is analytic in $|z| < R$ and $r(z)$ is a strictly proper rational function which is the sum of the principal parts of the Laurent series of $a(z)$ in neighborhoods of the poles of $a(z)$.

In [18] the problem was formulated of how the passage from the analytic function $b(z)$ to the meromorphic function $b(z) + r(z)$ influences the convergence of the Padé approximants. In [18,27] the convergence of the diagonal Padé approximants was studied for the case when $b(z)$ is the Markov function for a measure with a compact support on $\mathbb{R}$. Essential in the considerations of those works was the fact that the asymptotic behavior of denominators of the Padé approximants for $b(z)$ was well studied and $r(z)$ was considered as a kind of perturbation.

In contrast with [18,27] we will study the convergence of the row Padé approximants of $a(z)$ as the result of a small perturbation of the Padé approximants of the rational function $r(z)$ by $b(z)$. To do this, we will study the asymptotic behavior of the denominators of the Padé approximants for the rational function $r(z)$. Then considerations based on the stability allow to get the same asymptotic behavior for the Padé approximants of $a(z)$. Thus for the $(\lambda - 1)$th row the convergence of the Padé approximants of $a(z)$ is completely determined by the rational part $r(z)$ of the function $a(z)$. This approach was earlier used for a proof of a matrix analog of de Montessus's theorem [9].

2. Statement of main results

Let $a(z)$ be a function which is meromorphic in the disk D_R and analytic at the origin. Let $z_1, \dots, z_\ell$ be its distinct poles of multiplicities $s_1, \dots, s_\ell$, respectively, and let $\lambda = s_1 + \cdots + s_\ell$ be the number of its poles in the disk. In this article we will consider the Padé approximants $\pi_n(z) = \frac{P_n(z)}{Q_n(z)}$ of type $(n, \lambda - 1)$ for $a(z)$. In the sequel we will see that the asymptotic behavior of the denominators $Q_n(z)$ as $n \to \infty$ is defined by an arithmetic nature of these poles of the maximal modulus that have the maximal multiplicity. Let us introduce some needed notation.

Suppose that

$$\rho \equiv |z_1| = \cdots = |z_\mu| > |z_{\mu+1}| \geqslant \cdots \geqslant |z_\ell|.$$

The poles $z_1, \ldots, z_\mu$ of the maximal modulus are ordered in such a way that $s_1 \geqslant \cdots \geqslant s_\mu$. Among the poles $z_1, \ldots, z_\mu$ we select the poles $z_1, \ldots, z_v$ $(1 \leqslant v \leqslant \mu \leqslant \ell)$ that have the maximal multiplicity:

$$s \equiv s_1 = \cdots = s_v > s_{v+1} \geqslant \cdots \geqslant s_\mu.$$

The poles $z_1, \ldots, z_v$ will be called *dominant poles* of $a(z)$. Let

$$z_1 = \rho e^{2\pi i \Theta_1}, \ldots, z_v = \rho e^{2\pi i \Theta_v}.$$

Introduce the vector $\xi = (e^{2\pi i \Theta_1}, \ldots, e^{2\pi i \Theta_v})$ belonging to the torus $\mathbb{T}^v$. The asymptotic behavior of $Q_n(z)$ will be determined by the limiting behavior of ξ^n, $n \to \infty$. Hence we will need the closure $\mathbb{F}$ of the semigroup $\{\xi^n\}_{n \geqslant 0}$ in $\mathbb{T}^v$. This set coincides with the closure of the cyclic group $\{\xi^n\}_{n \in \mathbb{Z}}$ (see Theorem 5.1), i.e. $\mathbb{F}$ is a monothetic subgroup of the torus $\mathbb{T}^v$.

In the problem under consideration the group $\mathbb{F}$ plays a significant role. The limits of all convergent subsequences of $\{Q_n(z)\}$ form a family of polynomials which is parametrized by $\mathbb{F}$. For this reason we will call $\mathbb{F}$ by the *parameter group*.

Our first result is concerned with an explicit construction of $\mathbb{F}$ in terms of $\Theta_0 = 1, \Theta_1, \ldots, \Theta_v$. We will use the following well-known fact: an arbitrary integer matrix can be reduced to the diagonal form (the Smith form) by integer elementary transformations. Denote the field of rational numbers by $\mathbb{Q}$.

Theorem 2.1. *Let $r+1$ be the rank over $\mathbb{Q}$ of the system of the real numbers $\Theta_0 = 1, \Theta_1, \ldots, \Theta_v$.*

If $r = v$, then $\mathbb{F} = \mathbb{T}^v$.

Let $0 \leqslant r < v$. Put the poles $z_1, \ldots, z_v$ in such order that $\Theta_0 = 1, \Theta_1, \ldots, \Theta_r$ is a maximal linearly independent over $\mathbb{Q}$ subsystem of the system $\Theta_0, \Theta_1, \ldots, \Theta_v$, and

$$\Theta_j = \sum_{k=0}^{r} q_{kj} \Theta_k, \quad q_{kj} \in \mathbb{Q}, \ j = r+1, \ldots, v.$$

Let α_{kk} be the least common multiple of the denominators of the rational fractions q_{kj} and $\alpha_{kj} = \alpha_{kk} q_{kj}$ for $k = 0, \ldots, r$, $j = r+1, \ldots, v$. We compose the integer matrix

$$A = \begin{pmatrix} \alpha_{00} & \ldots & 0 & \alpha_{0,r+1} & \ldots & \alpha_{0v} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \ldots & \alpha_{rr} & \alpha_{r,r+1} & \ldots & \alpha_{rv} \end{pmatrix}$$

and reduce it to the Smith form over the ring $\mathbb{Z}$:

$$A = S_1 \Delta S_2^{-1}.$$

Here S_1, S_2 are invertible over $\mathbb{Z}$ integer matrices. Delete the first row and the first $r+1$ columns in S_2, and denote the obtained matrix by S. Let us now reduce S to the Smith form:

$$S = T_1 \Delta_0 T_2^{-1}.$$

Then the invariant factors of S are $1, \dots, 1, \sigma$, where σ is the greatest common divisor of minors of order $v-r$ of the matrix S.

Denote the $(v-r+j)$th row of T_1^{-1} by Q_j, $j = 0, \dots, r$. Then the group $\mathbb{F}$ consists of the points $\tau = (\tau_1, \dots, \tau_v) \in \mathbb{T}^v$ such that

$$\tau = \xi_n^{Q_0} t_1^{Q_1} \dots t_r^{Q_r}.$$

Here

$$\xi_n = e^{\frac{2\pi i n}{\sigma}}, \quad n = 0, \dots, \sigma - 1,$$

$(t_1, \dots, t_r)$ is an arbitrary point of the torus $\mathbb{T}^r$, and we use the notation $t^Q = (t^{q_1}, \dots, t^{q_v})$, where $Q = (q_1, \dots, q_v)$.

Now it is not difficult to prove that for any $\tau \in \mathbb{F}$ there exists the sequence $\Lambda_\tau \subseteq \mathbb{N}$ consisting of numbers $k_1, k_2, \dots,\ k_j < k_{j+1}$, such that

$$\lim_{k \to \infty} (e^{2\pi i k \Theta_1}, \dots, e^{2\pi i k \Theta_v}) = \tau, \quad k \in \Lambda_\tau.$$

We denote by $\Lambda_\tau - \lambda$ the sequence Λ_τ shifted by λ.

The following theorem is the main result of the work. In it we will describe the asymptotic behavior of the denominators $Q_n(z)$ and will calculate the limit points of the sequence $\{Q_n(z)\}$.

Let A_j be the coefficient of $(z-z_j)^{-s_j}$ in the Laurent series in a neighborhood of the pole $z = z_j$ for the function $a(z)$. Put

$$C_j = \frac{1}{(s_j - 1)! z_j^{s_j - 1} D_j^2(z_j) A_j},$$

where

$$D_j(z) = \frac{D(z)}{(z - z_j)^{s_j}}, \quad D(z) = (z - z_1)^{s_1} \cdots (z - z_\ell)^{s_\ell}, \quad 1 \leqslant l \leqslant \ell.$$

Denote

$$S_j(\tau) = \sum_{k=1}^{v} C_k z_k^j \tau_k, \quad \tau = (\tau_1, \dots, \tau_v) \in \mathbb{F},\ j \in \mathbb{Z}.$$

The nonnegative integer $\delta_+(\tau)$ $(\delta_-(\tau))$ will be called the *plus-defect* (*minus-defect*) of $\tau \in \mathbb{F}$ if $\delta_+(\tau)$ $(\delta_-(\tau))$ is the least nonnegative integer such that $S_{\delta_+(\tau)}(\tau) \neq 0$ $(S_{-\delta_-(\tau)-1}(\tau) \neq 0)$. It is easily seen that $0 \leqslant \delta_+(\tau) \leqslant v-1$, $1 \leqslant \delta_-(\tau) \leqslant v$. If τ is fixed, then for brevity we will write δ_+, δ_- instead of $\delta_+(\tau)$, $\delta_-(\tau)$. A polynomial $P(z)$ is said to be *d-normalized* if the coefficient of z^d in $P(z)$ is equal to 1.

Theorem 2.2. *Let τ be an arbitrary point of the group $\mathbb{F}$ and let Λ_τ be a sequence of indices corresponding to τ.*

Then, for all n, $n \in \Lambda_\tau - \lambda$, sufficiently large the denominator $Q_n(z)$ can be $(\lambda - \delta_+ - 1)$-normalized and for the sequence of the normalized polynomials $Q_n(z)$ there exists

$$\lim_{n\to\infty} Q_n(z) = W(z,\tau), \quad n \in \Lambda_\tau - \lambda, \ \tau \in \mathbb{F}.$$

Here $W(z,\tau)$ is a polynomial of degree $\lambda - \delta_+ - 1$ which has $z = 0$ as a root of multiplicity δ_-. This polynomial is computed by the formula:

$$W(z,\tau) = S_{\delta_+}^{-1}(\tau)\omega(z,\tau)(z - z_1)^{s_1 - 1}\dots(z - z_v)^{s_v - 1}(z - z_{v+1})^{s_{v+1}}\dots(z - z_\ell)^{s_\ell},$$

$$\omega(z,\tau) = \sum_{j=1}^{v} C_j \Delta_j(z)\tau_j, \quad \Delta_j = \frac{\Delta(z)}{z - z_j}, \quad \Delta(z) = (z - z_1)\dots(z - z_v).$$

The family $W(z,\tau)$, $\tau \in \mathbb{F}$, exhausts all limit points of the suitable normalized sequence $\{Q_n(z)\}$.

From this theorem it follows

Proposition 2.1. *For the denominators $Q_n(z)$ of the Padé approximants of type $(n, \lambda - 1)$ we have as $n \to \infty$, $n \in \Lambda_\tau - \lambda$, $\tau \in \mathbb{F}$:*

(1) *$s_i - 1$ roots of $Q_n(z)$ tend to z_i, for $i = 1, \dots, v$;*
(2) *s_i roots of $Q_n(z)$ tend to z_i for $i = v + 1, \dots, \ell$;*
(3) *$\delta_+(\tau)$ roots of $Q_n(z)$ tend to ∞, and $\delta_-(\tau)$ roots tend to* 0;
(4) *the other $v - \delta_+(\tau) - \delta_-(\tau) - 1$ roots of $Q_n(z)$ tend to the finite nonzero roots of $\omega(z,\tau)$ which are different from $z_1, \dots, z_v$.*

Hence the set of the limit points of poles of the sequence $\{\pi_n(z)\}$, $n \to \infty$, consists of the poles of $a(z)$ (the multiplicity of the poles $z_1, \dots, z_v$ is less by 1), possibly the points $z = 0$ and ∞, and the set of the zeros of polynomials from the family $\omega(z,\tau)$, $\tau \in \mathbb{F}$. The last set will be called the *set of additional limit points* and will be denoted by $\mathcal{N}_{\mathbb{F}}$.

The following theorem is also a direct consequence of the main result.

Theorem 2.3. *Fix $\tau \in \mathbb{F}$ and let Λ_τ be a corresponding sequence of numbers. Let K_τ be any compact set of the disk $|z| < \rho$ such that all zeros of the polynomial $W(z,\tau)$ from Theorem* 2.2 *lie outside K_τ.*

Then

$$\lim_{n\to\infty} ||\pi_n(z) - a(z)||_{C(K_\tau)} = 0, \quad n \in \Lambda_\tau - \lambda.$$

Now we state some results on the geometry of the set $\mathcal{N}_{\mathbb{F}}$. First we study the case when $\mathcal{N}_{\mathbb{F}} = \emptyset$. If $a(z)$ has a single dominant pole ($v = 1$), then it is easily seen that $\omega(z,\tau) = \text{const}$ and $\mathcal{N}_{\mathbb{F}} = \emptyset$. This means that the whole sequence $\pi_{n,\lambda-1}(z)$ has limit. We arrive at the result of Lin [22] and Sidi [29] mentioned in Introduction. It turns out that the converse statement is also true. Moreover, a stronger result holds. The polynomials $\omega(z,\tau)$ have the same set of zeros for all $\tau \in \mathbb{F}$ iff $\omega(z,\tau) = \text{const}$. Hence we obtain the following result.

Theorem 2.4. *The sequence of normalized denominators $Q_n(z)$ has limit as $n \to \infty$ if and only if $v = 1$.*

The next case is $v = 2$. The group $\mathbb{F}$ is easily computed and we obtain

Theorem 2.5. *Let $v = 2$, i.e. $a(z)$ has exactly two dominant poles z_1, z_2. If z_1, z_2 are vertices of a regular σ-gon, then $\mathcal{N}_{\mathbb{F}}$ consists of σ or $\sigma - 1$ points lying on the Apollonius circle*

$$\left|\frac{z - z_1}{z - z_2}\right| = \left|\frac{C_1}{C_2}\right|.$$

In the converse case the set $\mathcal{N}_{\mathbb{F}}$ coincides with this circle (or this straight line if $|\frac{C_1}{C_2}| = 1$).

If we put $\ell = \mu = v = 2,\ s = 1$, then we obtain the example of Liu [23, Section 5, Proposition 1].

Now let $v > 2$ and let $\mathcal{N}_j$ $(1 \leqslant j \leqslant v)$ be the set of complex points z satisfying the inequality

$$|C_j \Delta_j(z)| \leqslant \sum_{\substack{k=1 \\ k \neq j}}^{v} |C_k \Delta_k(z)|, \quad j = 1, \dots, v.$$

Put $\mathcal{N} = \bigcap_{j=1}^{v} \mathcal{N}_j$. It is not difficult to show that for $v > 2$ the set $\mathcal{N}_{\mathbb{F}}$ is a nonempty closed subset of $\mathcal{N}$.

Theorem 2.6. *If $r = v$, i.e. the numbers $\Theta_0 = 1, \Theta_1, \dots, \Theta_v$ are linearly independent over $\mathbb{Q}$, then*

$$\mathcal{N}_{\mathbb{F}} = \mathcal{N}.$$

In the following case the set $\mathcal{N}_{\mathbb{F}}$ consists of a finite number of points.

Theorem 2.7. *The set $\mathcal{N}_{\mathbb{F}}$ is finite if and only if the dominant poles $z_1, \dots, z_v$ are vertices of a regular σ-gon, $\sigma \geqslant v$. If this condition is fulfilled, then $\mathcal{N}_{\mathbb{F}}$ consists of the* *roots of the polynomials*

$$\omega_j(z) = \sum_{k=1}^{v} C_k \Delta_k(z) z_k^j, \quad j = 0, 1, \ldots, \sigma - 1.$$

These polynomials can be obtained by the recurrence formula:

$$\omega_{n+1}(z) = z\omega_n(z) - \alpha_n \Delta(z), \quad n \geqslant 0.$$

Here α_n *is the formal leading coefficient of* $\omega_n(z)$.

For other cases of explicit construction of the set $\mathcal{N}_{\mathbb{F}}$ see Section 9.

The results on the geometry of $\mathcal{N}_{\mathbb{F}}$ can be used in numerical analyses for localization of poles of meromorphic functions. This will be the subject of forthcoming paper.

3. Method of essential polynomials and Padé approximants

Our approach to a convergence problem for a row sequence of Padé approximants of meromorphic functions is based on a method of essential polynomials. Notions of indices and essential polynomials were first introduced in [1] in connection with an explicit construction of a Wiener–Hopf factorization for triangular 2×2 matrix functions (see also [2–4]). The algebraic results that derived by this method are summarized in [7]; the analytic applications to the Wiener–Hopf factorization of meromorphic matrix functions are given in [5,6,8]. Moreover, de Montessus's theorem for matrix Padé approximants was also proved on the basis of this approach [9].

In this section we will reformulate the classical definition of the Padé approximants in terms of essential polynomials. We begin with the definition that naturally leads to the notions of the essential polynomials. Since we will consider the row convergence, we can restrict ourselves to the case of the Padé approximant of type (n, m) for $n \geqslant m$.

Definition 3.1. Let $a(z) = \sum_{i=0}^{\infty} a_i z^i$ be a (formal) power series. The Padé approximant of type (n, m) and order k $(n - m \leqslant k \leqslant n + m)$ for $a(z)$ is a rational function $\pi_{n,m}^{(k)}(z) = P_{n,m}^{(k)}(z)/Q_{n,m}^{(k)}(z)$ such that the polynomials $P_{n,m}^{(k)}(z)$ and $Q_{n,m}^{(k)}(z)$ satisfy the following conditions:

1. $Q_{n,m}^{(k)}(z) \not\equiv 0$, $\deg Q_{n,m}^{(k)}(z) \leqslant k - n + m$,
2. $\deg P_{n,m}^{(k)}(z) \leqslant k$,
3. $a(z) Q_{n,m}^{(k)}(z) - P_{n,m}^{(k)}(z) = \mathcal{O}(z^{n+m+1})$, $z \to 0$.

If $k = n + m$, then for any polynomial $Q_{n,m}^{(n+m)}(z)$ of formal degree $2m$ Condition 3 is fulfilled automatically. Obviously, the classical definition of the Padé approximant of type (n, m) corresponds to our definition for $k = n$. We note that any Padé approximant of type (n, m) and order $k \leqslant n$ is the classical Padé approximants of type (n, m).

It is easy to verify that the vector consisting of the coefficients of the polynomial $Q_{n,m}^{(k)}(z)$ belongs to the kernel of the Toeplitz matrix

$$T_{k+1} = \begin{pmatrix} a_{k+1} & a_k & \dots & a_{n-m+1} \\ a_{k+2} & a_{k+1} & \dots & a_{n-m+2} \\ \vdots & \vdots & & \vdots \\ a_{n+m} & a_{n+m-1} & \dots & a_{2n-k} \end{pmatrix}$$

for $n - m \leqslant k \leqslant n + m - 1$. (We prefer to deal with Toeplitz matrices in place of often used Hankel ones.) Since the numerator $P_{n,m}^{(k)}(z)$ can be easily found by the denominator $Q_{n,m}^{(k)}(z)$, we concentrate on a determination of the polynomial $Q_{n,m}^{(k)}(z)$.

Hence we must study the kernel structure of matrices from the family $\{T_k\}_{k=n-m+1}^{n+m}$ (for detail we refer to the work [7]). By a_{n-m+1}^{n+m} we denote the sequence $\{a_{n-m+1}, a_{n-m+2}, \dots, a_{n+m}\}$ generating this family. If we want to emphasize the dependence of T_k on a_{n-m+1}^{n+m}, we will use the notation $T_k(a_{n-m+1}^{n+m})$. By $\ker A$ we denote the kernel of a matrix A and by $[A]_j$ ($[A]^j$) denote the jth row (the jth column) of A.

Since it is more convenient to deal not with the coefficients of polynomials but directly with the polynomials, we pass from the spaces $\ker T_k(a_{n-m+1}^{n+m})$ to the spaces $\mathcal{N}_k(a_{n-m+1}^{n+m})$ of generating polynomials. To do this, with the help of the series $a(z)$ we introduce on the space rational functions of the form $R(z) = \sum_{j=M}^{N} r_j z^j$ the Stieltjes functional σ by the formula

$$\sigma\{R(z)\} = \sum_{j=M}^{N} a_{-j} r_j.$$

Here we suppose that $a_i = 0$ for $i < 0$. It is easily seen that $\sigma\{z^{-i}R(z)\}$ coincides with the coefficient of z^i in the Laurent series $a(z)R(z)$.

By $\mathcal{N}_k(a_{n-m+1}^{n+m})$ $(n - m + 1 \leqslant k \leqslant n + m)$ we denote the space of polynomials of formal degree $k - n + m - 1$ satisfying the following orthogonality conditions:

$$\sigma\{z^{-i}R(z)\} = 0, \quad i = k, k+1, \dots, n+m. \tag{3.1}$$

For brevity, we will also use the notation $\mathcal{N}_k$. Obviously, $\mathcal{N}_k$ is the space of generating polynomials of vectors in $\ker T_k$. For convenience, we put $\mathcal{N}_{n-m} = 0$ and denote by $\mathcal{N}_{n+m+1}$ the $(2m+1)$-dimensional space of all polynomials of formal degree $2m$. Thus $Q_{n,m}^{(k)}(z) \in \mathcal{N}_k(a_{n-m+1}^{n+m})$. By d_k we denote the dimension of $\mathcal{N}_k$. Let $\Delta_k = d_k - d_{k-1}$, $n - m + 1 \leqslant k \leqslant n + m + 1$. If a_{n-m+1}^{n+m} is a nonzero sequence, then $\Delta_{n-m+1} = 0$, $\Delta_{n+m+1} = 2$. It follows from conditions (3.1) that $\mathcal{N}_k$ and $z\mathcal{N}_k$ are subspaces of $\mathcal{N}_{k+1}$ $(n-m+1 \leqslant k \leqslant n+m)$. Moreover, it is easy to see that

$$\mathcal{N}_k \cap z\mathcal{N}_k = z\mathcal{N}_{k-1}.$$

Therefore, by the Grassmann formula

$$\dim[\mathcal{N}_k + z\mathcal{N}_k] = 2d_k - d_{k-1}. \tag{3.2}$$

Let h_{k+1} be the dimension of any complement $\mathcal{H}_{k+1}$ of the subspace $\mathcal{N}_k + z\mathcal{N}_k$ in the whole space $\mathcal{N}_{k+1}$. Then $h_{k+1} = \Delta_{k+1} - \Delta_k$, i.e. $\Delta_{k+1} \geqslant \Delta_k$. Thus,

$$0 = \Delta_{n-m+1} \leqslant \Delta_{n-m} \leqslant \cdots \leqslant \Delta_{n+m} \leqslant \Delta_{n+m+1} = 2.$$

It follows from this that there are two integers μ_1, μ_2 $(n-m+1 \leqslant \mu_1 \leqslant \mu_2 \leqslant n+m)$ such that

$$\Delta_{n-m+1} = \cdots = \Delta_{\mu_1} = 0,$$

$$\Delta_{\mu_1+1} = \cdots = \Delta_{\mu_2} = 1,$$

$$\Delta_{\mu_2+1} = \cdots = \Delta_{n+m+1} = 2. \tag{3.3}$$

If the second row is absent, we assume that $\mu_1 = \mu_2$. It is easy to verify that $\mu_1 + \mu_2 = 2n+1$. Hence in our case $\mu_1 \leqslant n < \mu_2$.

Definition 3.2. The integers $\mu_1,\ \mu_2$ defined in Eqs. (3.3) will be called the indices of the sequence a_{n-m+1}^{n+m}.

We see from Eqs. (3.3) that $h_{k+1} \neq 0$ for $k = \mu_1$ and $k = \mu_2$ only. Moreover, since $\mu_1 < \mu_2$, we have $h_{\mu_1+1} = h_{\mu_2+1} = 1$. Hence,

$$\mathcal{N}_{k+1} = \mathcal{N}_k + z\mathcal{N}_k$$

for $k \neq \mu_i$, and

$$\mathcal{N}_{\mu_i+1} = (\mathcal{N}_{\mu_i} + z\mathcal{N}_{\mu_i}) \dot{+} \mathcal{H}_{\mu_i+1}, \quad i = 1, 2.$$

Here $\mathcal{H}_{\mu_i+1}$ is an one-dimensional subspace of $\mathcal{N}_{\mu_i+1}$.

Now we can describe the kernel structure of the family $\{T_k\}_{k=n-m+1}^{n+m}$. For $k \in [n-m+1; \mu_1]$ we have $\mathcal{N}_k = 0$. The first nonzero space is the one-dimensional space $\mathcal{N}_{\mu_1+1}$. Let $R_1(z)$ be its basis. Then for $k \in [\mu_1+1; \mu_2]$

$$\mathcal{N}_k = \{q_1(z)R_1(z)\},$$

where $q_1(z)$ is an arbitrary polynomial of formal degree $k - \mu_1 - 1$. In particular, $\mathcal{N}_{n+1} = \{q_1(z)R_1(z)\}$, where $\deg q_1(z) \leqslant n - \mu_1$, and we obtain the parametrization of the set of denominators for the classical Padé approximant of type (n, m).

Let $R_2(z)$ be a basis of any complement $\mathcal{H}_{\mu_2+1}$ of the subspace $\mathcal{N}_{\mu_2} + z\mathcal{N}_{\mu_2} = \{q_1(z)R_1(z)\}$ in the whole space $\mathcal{N}_{\mu_2+1}$. Then

$$\mathcal{N}_k = \{q_1(z)R_1(z) + q_2(z)R_2(z)\}$$

for $k \in [\mu_2+1; n+m+1]$. Here $q_i(z)$ is an arbitrary polynomial of formal degree $k - \mu_i - 1$, $i = 1, 2$.

Thus the polynomials $R_1(z),\ R_2(z)$ play an essential role in the description of the spaces $\mathcal{N}_k$. It is easily seen that if $\{c_k\}$ are moments of a measure with respect to the unit circle, then $R_1(z)$ for the sequence c_{-n}^{n-1} is the orthogonal polynomial of degree n on the unit circle.

Definition 3.3. The polynomials $R_1(z),\ R_2(z)$ will be called the essential polynomials of the sequence a_{n-m+1}^{n+m}.

Note that the first essential polynomial $R_1(z)$ is unique up to multiplication by a constant, while there is sufficient degree of freedom in a choice of the second essential polynomial $R_2(z)$.

In what follows we must be able to verify that the given integers μ_1, μ_2 $(n-m \leqslant \mu_1 \leqslant \mu_2 \leqslant n+m,\ \mu_1+\mu_2 = 2n+1)$ are indices and that the given polynomials $R_1(z) \in \mathcal{N}_{\mu_1+1},\ R_2(z) \in \mathcal{N}_{\mu_2+1}$ are essential polynomials of the sequence a_{n-m+1}^{n+m}.

It turns out that in order to verify essentialness we must test the inequality

$$\sigma_0 \equiv \sigma\{z^{-\mu_1} R_{2,\mu_2-n+m} R_1(z) - z^{-\mu_2} R_{1,\mu_1-n+m} R_2(z)\} \neq 0$$

(see [6,7] at once for the matrix case). Here R_{i,μ_i-n+m} is the formal leading coefficient of $R_i(z)$. The number σ_0 will be called the *test number* for the given integers μ_1, μ_2 and polynomials $R_1(z), R_2(z)$.

Now we find when the indices of a sequence a_{n-m+1}^{n+m} are stable under small perturbations. It is easily seen that if $k \leqslant \mu_1$, then T_k is left invertible and if $k \geqslant \mu_2$, then T_k is right invertible. Since the set of one-sided invertible operators is open, the indices $\bar{\mu}_1, \bar{\mu}_2$ of a perturbation sequence satisfy the inequality

$$\mu_1 \leqslant \bar{\mu}_1 \leqslant \bar{\mu}_2 \leqslant \mu_2.$$

Hence, if $\mu_2 - \mu_1 \leqslant 1$, i.e.,

$$\mu_1 = n, \quad \mu_2 = n+1,$$

then the indices μ_1, μ_2 are stable under small perturbations. It can be shown that this condition is also necessary for the stability of the indices. We note that if the Hadamard determinant

$$\Delta_{n+1,m} = \det ||a_{i-j}||_{\substack{i=n+1,\ldots,n+m \\ j=0,1,\ldots,m-1}}$$

is nonvanishing, then the indices are stable.

Return to the Padé approximants. After we study the kernel structure of the matrices $T_k(a_{n-m+1}^{n+m})$, the following definition will be natural.

Definition 3.4. The Padé approximant $\pi_{n,m}^{(\mu_1)}(z)$ of type (n,m) and minimal order $k = \mu_1$ will be called the Padé approximant $\pi_{n,m}(z)$ of type (n,m).

Summing the above considerations, we come to the following result.

Theorem 3.1. *The Padé approximant* $\pi_{n,m}(z) = \frac{P_{n,m}(z)}{Q_{n,m}(z)}$ *of type* (n, m) *always exists and has the minimal order. The numerator and the denominator of* $\pi_{n,m}(z)$ *are uniquely determined up to multiplication by a constant, and* $Q_{n,m}(z)$ *is the first essential polynomials* $R_1(z)$ *of the sequence* a_{n-m+1}^{n+m}. *Moreover,* $\pi_{n,m}(z)$ *is the classical Padé approximant and also it coincides with the Padé–Baker approximant if the last exists.*

The Padé–Baker approximant exists if and only if $R_1(0) \neq 0$. *If* $R_1(0) = 0$, *then the index of defect* $\omega_{n,m}$ *coincides with the multiplicity* $z = 0$ *as a root of* $R_1(z)$.

It is only required to prove the statements on the Padé–Baker approximant. But the proof is straightforward and therefore is omitted. Moreover, these statements are not used in the following.

Thus we clarify the place of the index μ_1 and the first essential polynomial $R_1(z)$ of the sequence a_{n-m+1}^{n+m} in the theory of the Padé approximants. As we will see in Section 8, in our approach the second essential polynomial $R_2(z)$ also plays an important role. For completeness we formulate yet another result that make clear the place of $R_2(z)$. This result is also not used in the given work.

Theorem 3.2. *Suppose the Hadamard determinant*

$$\Delta_{n,m} = \det ||a_{i-j}||_{\substack{i=n,\dots,n+m \\ j=0,1,\dots,m}}$$

is nonvanishing.

Then the sequences a_{n-m-1}^{n+m} *and* a_{n-m+1}^{n+m} *associated with the Padé approximants of type* $(n-1, m+1)$ *and* (n, m), *respectively, have the stable indices.*

Moreover, the first essential polynomial $R_1^{(n,m)}(z)$ *of the sequence* a_{n-m+1}^{n+m} *can be taken as the second essential polynomial* $R_2^{(n-1,m+1)}(z)$ *of the sequence* a_{n-m-1}^{n+m}.

Conversely, for the sequence a_{n-m-1}^{n+m} *there exists unique, up to a constant factor, the second essential polynomial* $R_2^{(n-1,m+1)}(z)$ *such that its degree is less than or equals* m. *This polynomial can be taken as the first essential polynomial* $R_1^{(n,m)}(z)$ *of* a_{n-m+1}^{n+m}.

In conclusion of the section we note that the determination of indices and essential polynomials of a sequence is equivalent to solving the problem of the Wiener–Hopf factorization for a triangular 2×2 matrix function (see [1,7]). Hence the Padé approximation problem is also equivalent to this factorization problem.

4. Explicit construction of denominators of Padé approximants for rational functions

Throughout what follows, we will consider the $(\lambda - 1)$th row of the Padé table and now we will omit the subscript $m = \lambda - 1$.

As we will show in Section 8, the study of a convergence of $Q_n(z)$ for a meromorphic function $a(z)$ is reduced to the same problem for the rational part $r(z)$ of $a(z)$. In this section we find the denominators $Q_n(z)$ for $r(z)$. By result of Section 3, we have to study the indices and essential polynomials of the sequence r_{n-m+1}^{n+m} for $m = \lambda - 1$.

First we describe in what terms we will solve this problem. Let $r(z)$ be a strictly proper rational function which is analytic at the origin. Let us represent it as the fraction of coprime polynomials $N(z)$ and $D(z)$:

$$r(z) = \frac{N(z)}{D(z)},$$

where $D(z) = (z - z_1)^{s_1} \cdots (z - z_\ell)^{s_\ell} = z^\lambda + d_{\lambda-1} z^{\lambda-1} + \cdots + d_0$ and $\deg N(z) < \lambda$. Since the polynomials $N(z)$ and $D(z)$ are coprime, there exist polynomials $U_0(z)$, $V_0(z)$ such that

$$U_0(z)D(z) + V_0(z)N(z) = 1. \tag{4.1}$$

This equation is called the *Bezout equation* and its solution $(U_0(z), V_0(z))$ can be found by the Euclidean algorithm. Moreover, it is well known that we can choose $U_0(z)$, $V_0(z)$ in such a way that $\deg V_0(z) < \lambda$. In this case the solution $(U_0(z), V_0(z))$ is unique and is called the *minimal solution* of the Bezout equation. (The terminology is borrowed from system theory, where the matrix Bezout equation is widely used.)

Now let us consider the equation

$$U_k(z)D(z) + V_k(z)N(z) = z^k, \quad k \geqslant 0. \tag{4.2}$$

This equation will be called the *Bezout equation of order* k. A solution of Eq. (4.2) such that $\deg V_k(z) < \lambda$ is said to be *minimal*.

Proposition 4.1. *For all* $k \geqslant 0$ *Eq.* (4.2) *has a unique minimal solution that can be found by the recurrence formula*

$$V_{k+1}(z) = zV_k(z) - v_k D(z), \quad U_{k+1}(z) = zU_k(z) + v_k N(z), \quad k \geqslant 0. \tag{4.3}$$

Here v_k *is the coefficient of* $z^{\lambda-1}$ *in* $V_k(z)$, *i.e. the formal leading coefficient of* $V_k(z)$. *Moreover, for* $k \geqslant 2\lambda - 1$ *degree of the polynomial* $U_k(z)$ *is* $k - \lambda$ *and the leading coefficient of this polynomial equals* 1.

Proof. Let $U_k(z)$, $V_k(z)$ be the polynomials determined by Eq. (4.3) and the initial minimal solution $(U_0(z), V_0(z))$. Obviously, $\deg V_k(z) \leqslant \lambda - 1$ and $(U_k(z), V_k(z))$ is a solution of Eq. (4.2). Hence a minimal solution exists.

Assume now that there exists another minimal solution $(\tilde{U}_k(z), \tilde{V}_k(z))$. Then

$$(U_k(z) - \tilde{U}_k(z))D(z) = (\tilde{V}_k(z) - V_k(z))N(z).$$

Denote $W(z) = [\tilde{V}_k(z) - V_k(z)]D^{-1}(z)$. Hence,

$$U_k(z) - \tilde{U}_k(z) = W(z)N(z), \quad \tilde{V}_k(z) - V_k(z) = W(z)D(z).$$

Multiplying these equations by $V_0(z)$ and $U_0(z)$, respectively, and summing, we obtain

$$W(z) = [U_k(z) - \tilde{U}_k(z)]V_0(z) + [\tilde{V}_k(z) - V_k(z)]U_0(z),$$

i.e. $W(z)$ is a polynomial. But since $\deg[\tilde{V}_k(z) - V_k(z)] < \deg D(z) = \lambda$, the equality $\tilde{V}_k(z) - V_k(z) = W(z)D(z)$ is possible iff $W(z) \equiv 0$. Hence $\tilde{V}_k(z) = V_k(z)$, $\tilde{U}_k(z) = U_k(z)$.

It remains to prove the last statement of the proposition. Let $k \geqslant 2\lambda - 1$. Since $\deg V_k(z)N(z) \leqslant 2\lambda - 2$, then $\deg U_k(z)D(z) = k$. Hence $\deg U_k(z) = k - \lambda$, and the leading coefficient of $U_k(z)$ is 1. This completes the proof. $\square$

Now we can obtain the indices and essential polynomials of the sequence $r_{n-\lambda+2}^{n+\lambda-1} = \{r_{n-\lambda+2}, r_{n-\lambda+3}, \dots, r_{n+\lambda-1}\}$ consisting of the Taylor coefficients of a rational function $r(z)$. This sequence is required for a determination of the denominators $Q_n(z)$ of the Padé approximant of type $(n, \lambda - 1)$ for $r(z)$.

Theorem 4.1. *For $n \geqslant \lambda$ the sequence $r_{n-\lambda+2}^{n+\lambda-1}$ has the stable indices $n, n+1$ and the essential polynomials $R_1(z) = V_{n+\lambda}(z)$, $R_2(z) = D(z)$. Moreover, the test number of $R_1(z)$, $R_2(z)$ is $\sigma_0 = -1$.*

Proof. First we verify that

$$V_{n+\lambda}(z) \in \mathcal{N}_{n+1}(r_{n-\lambda+2}^{n+\lambda-1}), \quad D(z) \in \mathcal{N}_{n+2}(r_{n-\lambda+2}^{n+\lambda-1}).$$

From the Bezout equation (4.2) we have

$$r(z)V_{n+\lambda}(z) + U_{n+\lambda}(z) = z^{n+\lambda}D^{-1}(z) = \mathcal{O}(z^{n+\lambda}), \quad z \to 0. \tag{4.4}$$

Since $(U_{n+\lambda}(z), V_{n+\lambda}(z))$ is the minimal solution, from Proposition 4.1 we get

$$\deg V_{n+\lambda}(z) \leqslant \lambda - 1, \quad \deg U_{n+\lambda}(z) = n.$$

Hence $V_{n+\lambda}(z)$ is the denominator of the classical Padé approximant $\pi_n(z)$ of type $(n, \lambda - 1)$ for $r(z)$. Thus $V_{n+\lambda}(z) \in \mathcal{N}_{n+1}(r_{n-\lambda+2}^{n+\lambda-1})$.

Let us prove the second inclusion. Polynomials in $\mathcal{N}_{n+2}(r_{n-\lambda+2}^{n+\lambda-1})$ have formal degree λ. Degree of $D(z)$ is also λ. It follows from the equality

$$r(z)D(z) = N(z)$$

that the coefficients of $z^{\lambda}, z^{\lambda+1}, \dots$ in the power series $r(z)D(z)$ are equal to zero because $\deg N(z) \leqslant \lambda - 1$. This means that

$$\sigma\{z^{-i}D(z)\} = 0, \quad i = \lambda, \lambda + 1, \dots . \tag{4.5}$$

For a polynomial $R(z)$ in $\mathcal{N}_{n+2}(r_{n-\lambda+2}^{n+\lambda-1})$ the following orthogonality conditions

$$\sigma\{z^{-i}R(z)\} = 0, \quad i = n+2, n+3, \dots, n+\lambda-1.$$

must be fulfilled. Therefore for $n \geqslant \lambda$ the polynomial $D(z)$ really belongs to the space $\mathcal{N}_{n+2}(r_{n-\lambda+2}^{n+\lambda-1})$. We note that if $\lambda = 2$, i.e. if $r_n^{n+1} = \{r_n, r_{n+1}\}$, then the orthogonality conditions are absent, and the space $\mathcal{N}_{n+2}(r_n^{n+1})$ coincides with the space of all polynomials of formal degree 2 (see Section 3).

Now we put $\mu_1 = n$, $\mu_2 = n+1$, $R_1(z) = V_{n+\lambda}(z)$, $R_2(z) = D(z)$ and calculate the test number σ_0. In our case

$$\sigma_0 = \sigma\{z^{-n}V_{n+\lambda}(z)\} - v_{n+\lambda}\sigma\{z^{-n-1}D(z)\},$$

where $v_{n+\lambda}$ is the (formal) leading coefficient of $V_{n+\lambda}(z)$. For $n \geqslant \lambda$, by virtue of Eq. (4.5), $\sigma\{z^{-n-1}D(z)\} = 0$. The number $\sigma\{z^{-n}V_{n+\lambda}(z)\}$ is the coefficient of z^n in the power series $r(z)V_{n+\lambda}(z)$. From (4.4) we see that this coefficient coincides with the leading coefficient of $U_{n+\lambda}(z)$ with the opposite sign. Hence $\sigma_0 = -1$. Then $n, n+1$ are indices and $V_{n+\lambda}(z)$, $D(z)$ are the essential polynomials of the sequence $r_{n-\lambda+2}^{n+\lambda-1}$. $\square$

Thus we obtain the sequence of denominators $Q_n(z) = V_{n+\lambda}(z)$. Now we find an explicit formula for $V_k(z)$ in terms of v_k.

Theorem 4.2. *The polynomials $V_k(z)$ satisfy the following linear difference equation*

$$V_{k+\lambda}(z) + d_{\lambda-1}V_{k+\lambda-1}(z) + \cdots + d_0 V_k(z) = 0, \quad k \geqslant 0, \tag{4.6}$$

where $D(z) = z^\lambda + d_{\lambda-1}z^{\lambda-1} + \cdots + d_0$. Moreover,

$$V_k(z) = \sum_{j=1}^{\lambda} \sum_{i=1}^{j} d_{\lambda-i+1} v_{k+j-i} z^{\lambda-j}, \quad k \geqslant 0. \tag{4.7}$$

Proof. From the recurrence formula

$$V_{k+1}(z) = zV_k(z) - v_k D(z), \quad k \geqslant 0,$$

it follows that

$$V_{k+i}(z) = z^i V_k(z) - \sum_{j=0}^{i-1} v_{k+i-j-1} z^j D(z), \quad i \geqslant 1.$$

Multiplying this equation by d_i and summing by i from 0 to λ, we get

$$\sum_{i=0}^{\lambda} d_i V_{k+i}(z) = D(z)\left[V_k(z) - \sum_{i=1}^{\lambda} \sum_{j=0}^{i-1} d_i v_{k+i-j-1} z^j\right].$$

Suppose that the polynomial in the square brackets is nonzero. Then degree of the left-hand side of the above equality is less than or equals $\lambda - 1$, while the right-hand side has degree $\geqslant \lambda$. Since this is impossible, we obtain

$$\sum_{i=0}^{\lambda} d_i V_{k+i}(z) = 0$$

and

$$V_k(z) = \sum_{i=1}^{\lambda} \sum_{j=0}^{i-1} d_i v_{k+i-j-1} z^j. \tag{4.8}$$

After the inversion of the order of summation and change of the index of summation in the last equation, we arrive at formula (4.7). $\square$

To study an asymptotic behavior of $V_{n+\lambda}(z)$ we must investigate a behavior of the leading coefficients v_k as $k\to\infty$. Comparison of the coefficients of $z^{\lambda-1}$ in Eq. (4.6) yields

$$v_{k+\lambda}+d_{\lambda-1}v_{k+\lambda-1}+\cdots+d_0v_k=0,\quad k\geqslant 0. \tag{4.9}$$

Thus the sequence of the leading coefficients v_k of the polynomials $V_k(z)$ is also a solution of a linear difference equation. By the theorem on the structure of the general solution of a linear difference equation with constant coefficients (see, e.g., [16]), we obtain

$$v_k=p_1(k)z_1^k+\cdots+p_\ell(k)z_\ell^k,\quad k\geqslant 0. \tag{4.10}$$

Here $z_1,\dots,z_\ell$ are the distinct zeros of $D(z)=(z-z_1)^{s_1}\cdots(z-z_\ell)^{s_\ell}$, and $p_j(k)=C_j^0+C_j^1k+\cdots+C_j^{s_j-1}k^{s_j-1}$ is a polynomial in k of formal degree s_j-1.

It remains to determine the polynomials $p_j(k)$ via the initial polynomial $V_0(z)$.

First we will find initial value $v_0,v_1,\dots,v_{\lambda-1}$ for Eq. (4.9) via the coefficients of the polynomial $V_0(z)$. Let $V_0(z)=\alpha_0z^{\lambda-1}+\alpha_1z^{\lambda-2}+\cdots+\alpha_{\lambda-1}$. On the other hand, by Theorem 4.2,

$$V_0(z)=v_0z^{\lambda-1}+(v_1+d_{\lambda-1}v_0)z^{\lambda-2}+\cdots+(v_{\lambda-1}+d_1v_{\lambda-2}+\cdots+d_1v_0).$$

Equate the coefficients of the same degrees of z. This yields the system of equations from which we obtain

$$\begin{pmatrix} v_0\\ v_1\\ \vdots\\ v_{\lambda-1}\end{pmatrix}=T^{-1}\begin{pmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{\lambda-1}\end{pmatrix}, \tag{4.11}$$

where

$$T=\begin{pmatrix} 1 & 0 & \dots & 0\\ d_{\lambda-1} & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ d_1 & d_2 & \dots & 1\end{pmatrix}.$$

To find the coefficients of the polynomials $p_j(z)$ by the initial conditions it is necessary to solve the system

$$\begin{aligned} &p_1(0)+p_2(0)+\cdots+p_\ell(0)=v_0\\ &p_1(1)z_1+p_2(1)z_2+\cdots+p_\ell(1)z_\ell=v_1\\ &\dots\\ &p_1(\lambda-1)z_1^{\lambda-1}+p_2(\lambda-1)z_2^{\lambda-1}+\cdots+p_\ell(\lambda-1)z_\ell^{\lambda-1}=v_{\lambda-1}.\end{aligned} \tag{4.12}$$

The matrix of this system is the column generalized Vandermonde matrix W_{c} determined by the polynomial $D(z)$ (see, e.g., [15]). The structure of W_{c} can be described as follows. Let us introduce the block column $[W_{\mathrm{c}}]^j$ corresponding to the zero z_j of the polynomial $D(z)$. This column is the $\lambda \times s_j$ matrix

$$[W_c]^j = \begin{pmatrix} 1 & 0 & \dots & 0 \\ z_j & z_j & \dots & z_j \\ z_j^2 & 2z_j^2 & \dots & 2^{s_j-1}z_j^2 \\ & & \dots & \\ z_j^{\lambda-1} & (\lambda-1)z_j^{\lambda-1} & \dots & (\lambda-1)^{s_j-1}z_j^{\lambda-1} \end{pmatrix}.$$

Then

$$W = ([W_c]^1 \quad [W_c]^2 \quad \dots \quad [W_c]^\ell).$$

It is known that W_c is invertible [15].

In an analogous manner we define the row generalized Vandermonde matrix W_r:

$$W_r = \begin{pmatrix} [W_r]_1 \\ \vdots \\ [W_r]_\ell \end{pmatrix},$$

where

$$[W_r]_j = \begin{pmatrix} (\lambda-1)^{s_j-1}z_j^{\lambda-1} & \dots & 2^{s_j-1}z_j^2 & z_j & 0 \\ & \dots & & & \\ (\lambda-1)z_j^{\lambda-1} & \dots & 2z_j^2 & z_j & 0 \\ z_j^{\lambda-1} & \dots & z_j^2 & z_j & 1 \end{pmatrix}.$$

Now if we denote by p_j the column consisting of the coefficients of the polynomial $p_j(k)$, then the solution of system (4.12) is

$$\begin{pmatrix} p_1 \\ \vdots \\ p_\ell \end{pmatrix} = W_c^{-1} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\lambda-1} \end{pmatrix}. \tag{4.13}$$

Thus we obtain the polynomial $p_j(k)$ via $V_0(z)$. Hence we have explicitly constructed $V_k(z)$ via $V_0(z)$.

As we will see in Section 6, the asymptotic behavior of v_k is determined by the leading coefficients $C_j^{s_j-1}$ of the polynomials $p_j(k)$ for $j = 1, \dots, v$. For brevity, we will write C_j instead of $C_j^{s_j-1}$. Let us calculate C_j and prove the important inequalities $C_j \neq 0$ for all $j = 1, \dots, \ell$. To do this, at first we introduce the operator ∂ on the set of polynomials $X(z) = x_0 + x_1 z + \dots + x_n z^n$ by the formula

$$\partial X(z) = zX'(z).$$

It is easily seen that ∂ possesses all properties of the operator of differentiation and

$$\partial^k X(z) = x_1 z + 2^k x_2 z^2 + \dots + n^k x_n z^n, \quad k \geqslant 0.$$

Moreover, the polynomials $\partial^k X(z)$ can be used for the definition of multiplicity of a nonzero root of $X(z)$ instead of the derivatives $X^{(k)}(z)$.

Lemma 4.1. *Let $z_0 \neq 0$. Then $z = z_0$ is a root of $X(z)$ of multiplicity r if and only if*

$$X(z_0) = \partial X(z_0) = \cdots = \partial^{r-1} X(z_0) = 0, \quad \partial^r X(z_0) \neq 0.$$

The proof is trivial because the polynomials $\partial^k X(z)$ are linearly expressed in terms of $zX'(z), \ldots, z^k X^{(k)}(z)$:

$$\partial^k X(z) = \sum_{i=1}^{k} \alpha_{ik} z^i X^{(i)}(z),$$

where $\alpha_{kk} = 1$, and conversely

$$X^{(k)}(z) = z^{-k} \sum_{i=1}^{k} \beta_{ik} \partial^i X(z).$$

Theorem 4.3. *Let A_j be the coefficient of $(z - z_0)^{-s_j}$ in the Laurent series in a neighborhood of the pole $z = z_j$ for the function $a(z)$.*

Then

$$C_j = \frac{1}{(s_j - 1)! z_j^{s_j - 1} D_j^2(z_j) A_j}, \tag{4.14}$$

where $D_j(z) = \frac{D(z)}{(z - z_j)^{s_j}}$, $1 \leqslant j \leqslant \ell$.

Proof. It follows from (4.11), (4.13) that

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{\lambda-1} \end{pmatrix} = T W_{\mathrm{c}} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_\ell \end{pmatrix}. \tag{4.15}$$

Multiply this equation by the row generalized Vandermonde matrix W_{r}. Since

$$[W_{\mathrm{r}}]_j \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{\lambda-1} \end{pmatrix} = \begin{pmatrix} \partial^{s_j - 1} V_0(z_j) \\ \vdots \\ \partial V_0(z_j) \\ V_0(z_j) \end{pmatrix}, \tag{4.16}$$

we obtain

$$\begin{pmatrix} \mathscr{V}_0(z_1) \\ \vdots \\ \mathscr{V}_0(z_\ell) \end{pmatrix} = W_{\mathrm{r}} T W_{\mathrm{c}} \begin{pmatrix} p_1 \\ \vdots \\ p_\ell \end{pmatrix}.$$

Here $\mathscr{V}_0(z_j)$ is the vector in the right-hand side of Eq. (4.16). From this formula we get

$$V_0(z_j) = [W_{\mathrm{r}} T W_{\mathrm{c}}]_{N_j} \begin{pmatrix} p_1 \\ \vdots \\ p_\ell \end{pmatrix}, \tag{4.17}$$

where $[W_{\mathrm{r}} T W_{\mathrm{c}}]_{N_j}$ is the N_jth row of $W_{\mathrm{r}} T W_{\mathrm{c}}$ and $N_j = s_1 + \cdots + s_j$.

Let us find the row

$$[W_{\mathrm{r}} T W_{\mathrm{c}}]_{N_j} = [W_{\mathrm{r}}]_{N_j} T W_{\mathrm{c}} = (z_j^{\lambda-1} \quad \dots \quad z_1 \quad 1) T W_{\mathrm{c}}.$$

Denote

$$(z_j^{\lambda-1} \quad \dots \quad z_1 \quad 1) T = (h_1 \quad h_2 \quad \dots \quad h_\lambda),$$

where $h_k = \sum_{n=k}^{\lambda} d_n z_j^{n-k}$. Let $M_j^i(z)$ be the product of this row multiplied by the column

$$\begin{pmatrix} 0 \\ z \\ 2^i z^2 \\ \vdots \\ (\lambda-1)^i z^{\lambda-1} \end{pmatrix}.$$

Then

$$M_j^i(z) = \sum_{k=2}^{\lambda} h_k (k-1)^i z^{k-1}.$$

If we substitute h_k in the above formula and invert the order of summation, we obtain

$$\begin{aligned} M_j^i(z) &= \sum_{n=2}^{\lambda} d_n \sum_{k=2}^{n} (k-1)^i z_j^{n-k} z^{k-1} = \sum_{n=2}^{\lambda} d_n \partial^i \sum_{k=2}^{n} z_j^{n-k} z^{k-1} \\ &= \partial^i \sum_{n=2}^{\lambda} d_n \sum_{k=2}^{n} z_j^{n-k} z^{k-1}. \end{aligned}$$

Since $\sum_{k=2}^{n} z_j^{n-k} z^{k-1} = z\frac{z^{n-1}-z_j^{n-1}}{z-z_j}$, we have

$$\begin{aligned}
M_j^i(z) &= \partial^i\left[\frac{1}{z-z_j}\left(\sum_{n=2}^{\lambda} d_n z^n - \frac{z}{z_j}\sum_{n=2}^{\lambda} d_n z_j^n\right)\right] \\
&= \partial^i\left[\frac{1}{z-z_j}\left(D(z) - d_1 z - d_0 - \frac{z}{z_j}(D(z_j) - d_1 z_j - d_0)\right)\right] \\
&= \partial^i\left[\frac{D(z)}{z-z_j} + \frac{d_0}{z_j}\right] = \partial^i \frac{D(z)}{z-z_j}.
\end{aligned}$$

The polynomial $\frac{D(z)}{z-z_j}$ has the roots $z = z_k$ $(k \neq j, 1 \leqslant k \leqslant \ell)$ of multiplicity s_k and $z = z_j$ of multiplicity $s_j - 1$. Hence, by Lemma 4.1, we get

$$M_j^i(z_k) = 0, \quad k \neq j,\ 1 \leqslant k \leqslant \ell,\ i = 1, 2, \ldots, s_k - 1$$

and

$$M_j^i(z_j) = 0, \quad i = 1, 2, \ldots, s_j - 2,$$

$$M_j^{s_j-1}(z_j) = \partial^{s_j-1} \frac{D(z)}{z-z_j}\bigg|_{z=z_j} = z_j^{s_j-1}\left(\frac{D(z)}{z-z_j}\right)^{(s_j-1)}\Bigg|_{z=z_j} \neq 0.$$

Let us find the last derivative. Since

$$\frac{D(z)}{z-z_j} = (z-z_j)^{s_j-1} D_j(z),$$

by the Leibniz rule, we have

$$\left(\frac{D(z)}{z-z_j}\right)^{(s_j-1)} = \sum_{k=0}^{s_j-1} \binom{s_j-1}{k} ((z-z_j)^{s_j-1})^{(k)} D_j^{(s_j-k-1)}(z).$$

Therefore

$$\left(\frac{D(z)}{z-z_j}\right)^{(s_j-1)}\Bigg|_{z=z_j} = (s_j-1)! D_j(z_j).$$

Thus

$$M_j^{s_j-1}(z_j) = (s_j-1)! z_j^{s_j-1} D_j(z_j).$$

It is not difficult to verify that the formulas for $M_j^i(z)$ hold also for $i = 0$.

The result is

$$[W_{\mathrm{r}}TW_{\mathrm{c}}]_{N_j} = (0 \quad \dots \quad 0 \quad (s_j-1)!z_j^{s_j-1}D_j(z_j) \quad 0 \quad \dots \quad 0),$$

where nonvanishing element has the number N_j. From Eq. (4.17) we obtain

$$V_0(z_j) = (s_j-1)!z_j^{s_j-1}D_j(z_j)C_j, \quad j=1,\dots,\ell.$$

It follows from the Bezout equation that

$$V_0(z_j) = \frac{1}{N(z_j)}.$$

Hence

$$C_j = \frac{1}{(s_j-1)!z_j^{s_j-1}D_j(z_j)N(z_j)}, \quad j=1,\dots,\ell.$$

Let us decompose the strictly proper rational function $r(z)=\frac{N(z)}{D(z)}$ into the sum of partial rational fractions, and let A_j be the coefficient of $(z-z_j)^{-s_j}$ in this decomposition. Then $N(z_j)=A_jD_j(z_j)$ and we arrive at the final result

$$C_j = \frac{1}{(s_j-1)!z_j^{s_j-1}D_j^2(z_j)A_j}, \quad j=1,\dots,\ell.$$

The theorem is proved. □

As an evident consequence of this theorem we note that all coefficients C_j are nonzero.

5. Explicit construction of parameter group

In this section we prove Theorem 2.1 on the monothetic subgroup $\mathbb{F}$ of the torus $\mathbb{T}^v$.

In order to obtain the asymptotic of the minimal solution $V_k(z)$ as $k\to\infty$, we must study the asymptotic of the solution v_k of linear difference equation (4.9). This solution is of the form

$$v_k = p_1(k)z_1^k + \cdots + p_\ell(k)z_\ell^k,$$

and its asymptotic will be defined by the limiting behavior of ξ^k, $k\to\infty$. Here $z_1=\rho e^{2\pi i\Theta_1},\dots,z_v=\rho e^{2\pi i\Theta_v}$ and $\xi=(e^{2\pi i\Theta_1},\dots,e^{2\pi i\Theta_v})\in\mathbb{T}^v$. Hence we need the closure $\mathbb{F}$ of the cyclic group $\{\xi^k\}_{k\in\mathbb{Z}}$. $\mathbb{F}$ is a monothetic subgroup of the torus $\mathbb{T}^v$.

We will use the standard group-theoretic technique that is applied for a proof of the Kronecker theorem on Diophantine approximations (see, e.g., [13,26]).

Proof of Theorem 2.1. The group $\mathbb{T}$ (the unit circle in complex plane $\mathbb{C}$) is identified with $\mathbb{R}/\mathbb{Z}$. Let us introduce the subgroup G of the group $\mathbb{R}^v$ generated by the point $\Theta=(\Theta_1,\dots,\Theta_v)\in\mathbb{R}^v$ and the vectors $e_1,\dots,e_v$ forming the standard basis of $\mathbb{R}^\nu$:

$$G = \{m_1 e_1 + \cdots + m_\nu e_\nu + m\Theta\}_{(m_1,\dots,m_\nu,m)\in\mathbb{Z}^{\nu+1}}$$
$$= \{(m_1 + m\Theta_1, \dots, m_\nu + m\Theta_\nu)\}_{(m_1,\dots,m_\nu,m)\in\mathbb{Z}^{\nu+1}}.$$

Then the group $G/\mathbb{Z}^\nu$ is isomorphic to the cyclic group $\{n\Theta\}_{n\in\mathbb{Z}}$ generated by Θ.

Now the group $\mathbb{F}$ may be identified with the group $\overline{(G/\mathbb{Z}^\nu)} = \bar{G}/\mathbb{Z}^\nu$. Here $\bar{G}$ is the closure of the group G in $\mathbb{R}^\nu$. It is well known (see, e.g., [26, Proposition 40]) that $\bar{G} = (G^\perp)^\perp$, where $G^\perp$ is the annihilator of G in $\mathbb{R}^\nu$, i.e. a subgroup of the dual group $\hat{\mathbb{R}}^\nu$ consisting of characters $\chi\in\hat{\mathbb{R}}^\nu$ such that $\chi(g) = 0$ for all $g = (g_1, \dots, g_\nu)\in G$. Recall that we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ and therefore the identity element of $\mathbb{T}$ is the coset of 0 in $\mathbb{R}/\mathbb{Z}$. Moreover, the dual group $\hat{\mathbb{R}}^\nu$ is identified in a natural way with $\mathbb{R}^\nu$. Thus,

$$G^\perp = \left\{(x_1, \dots, x_\nu)\in\mathbb{R}^\nu \;\middle|\; \sum_{j=1}^{\nu} x_j g_j\in\mathbb{Z},\ g\in G\right\}.$$

From the definition of G it is easily seen that $G^\perp$ is a subgroup of $\mathbb{Z}^\nu$ and

$$G^\perp = \left\{(h_1, \dots, h_\nu)\in\mathbb{Z}^\nu \;\middle|\; \sum_{j=1}^{\nu} h_j\Theta_j\in\mathbb{Z}\right\}.$$

Now we construct a basis of $G^\perp$. Denote $\Theta_0 = 1$ and introduce the subgroup L of $\mathbb{Z}^{\nu+1}$:

$$L = \left\{(h_0, \dots, h_\nu)\in\mathbb{Z}^{\nu+1} \;\middle|\; \sum_{j=0}^{\nu} h_j\Theta_j = 0\right\}.$$

Then $G^\perp = \operatorname{pr} L$, where $\operatorname{pr}(h_0, \dots, h_\nu) = (h_1, \dots, h_\nu)$.

Put the poles $z_1, \dots, z_\nu$ in such order that $\Theta_0 = 1, \Theta_1, \dots, \Theta_r$ $(0\leqslant r\leqslant \nu)$ is a maximal linearly independent over $\mathbb{Q}$ subsystem of the system $\Theta_0, \Theta_1, \dots, \Theta_\nu$. If $r = \nu$, then, obviously, $G^\perp = 0$. Hence $\bar{G} = (G^\perp)^\perp = \mathbb{R}^\nu$ and $\mathbb{F} = \mathbb{T}^\nu$. Let $r<\nu$ and

$$\Theta_j = \sum_{k=0}^{r} q_{kj}\Theta_k, \quad q_{kj}\in\mathbb{Q},\ j = r+1, \dots, \nu.$$

Taking into account the linear independence over $\mathbb{Q}$ of $\Theta_0, \Theta_1, \dots, \Theta_r$, we see that in order to find L it is necessary to obtain integer solutions of the system

$$h_j + \sum_{k=r+1}^{\nu} q_{jk}h_k = 0, \quad j = 0, 1, \dots, r. \tag{5.1}$$

Let α_{jj} be the least common multiple of the denominators of the rational fractions $q_{j,r+1}, \dots, q_{j,\nu}$ and $\alpha_{jk} = \alpha_{jj}q_{jk}$ for $j = 0, \dots, r$, $k = r+1, \dots, \nu$. Then system (5.1) can be rewritten in the integer form

$$\alpha_{jj}h_j + \sum_{k=r+1}^{\nu} \alpha_{jk}h_k = 0, \quad j = 0, 1, \dots, r. \tag{5.2}$$

Denote by A the integer matrix

$$\begin{pmatrix} \alpha_{00} & \dots & 0 & \alpha_{0,r+1} & \dots & \alpha_{0v} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \alpha_{rr} & \alpha_{r,r+1} & \dots & \alpha_{rv} \end{pmatrix}$$

of this system. Obviously, rank $A = r + 1$. To solve system (5.2) we reduce A to the Smith form over the ring $\mathbb{Z}$:

$$A = S_1 \Delta S_2^{-1},$$

where S_1, S_2 are invertible over $\mathbb{Z}$ integer matrices and

$$\Delta = \begin{pmatrix} \alpha_0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \alpha_r & 0 & \cdots & 0 \end{pmatrix}.$$

We note that the integers $\alpha_0, \dots, \alpha_r$ are nonzero. Hence for a solution $\tilde{h} = (h_0, \dots, h_v)^t$ of system (5.2) we have $(S_2^{-1}\tilde{h})_0 = \cdots = (S_2^{-1}\tilde{h})_r = 0$. In addition, the elements of $(S_2^{-1}\tilde{h})_{r+1} = \cdots = (S_2^{-1}\tilde{h})_v$ of the column $S_2^{-1}\tilde{h}$ are integers. Thus the solution $\tilde{h}$ of system (5.2) has the form

$$\tilde{h} = S_2 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ k_1 \\ \vdots \\ k_{v-r} \end{pmatrix},$$

where $k_1, \dots, k_{v-r}$ are arbitrary integers, and any element $\tilde{h}$ of the group L can be represented in the form

$$\tilde{h} = k_1 [S_2]^{r+2} + \cdots + k_{v-r} [S_2]^{v+1}.$$

Recall that $[A]^j$ is the jth column of a matrix A. Since the columns of the matrix S_2 are linearly independent over $\mathbb{Z}$, $[S_2]^{r+2}, \dots, [S_2]^{v+1}$ is a basis of the group L. Hence to construct the group $G = \operatorname{pr} L$ we need the last v rows and the last $v - r$ columns of the matrix S_2. Let S be the integer $v \times (v - r)$ matrix obtained from S_2 by deleting of the first row and the first $r + 1$ columns. Then any element $h = \operatorname{pr} \tilde{h}$ of the group $G^{\perp}$ can be written as follows:

$$h = k_1 [S]^1 + \cdots + k_{v-r} [S]^{v-r},$$

i.e. $\{[S]^1, \dots, [S]^{\nu-r}\}$ is a generating system of the group $G^{\perp}$. Suppose that $n_1[S]^1 + \cdots + n_{\nu-r}[S]^{\nu-r} = 0$, where $n_j \in \mathbb{Z}$. Then

$$\begin{pmatrix} m \\ 0 \\ \vdots \\ 0 \end{pmatrix} = n_1[S_2]^{r+2} + \cdots + n_{\nu-r}[S_2]^{\nu+1} \in L,$$

i.e. $m = 0$. Thus $n_1 = \cdots = n_{\nu-r} = 0$, and $[S]^1, \dots, [S]^{\nu-r}$ is a basis of $G^{\perp}$.

However in order to obtain $\bar{G}$ we need a special basis of $G^{\perp}$. To construct it, we reduce the integer matrix S of rank $\nu - r$ to the Smith form:

$$S = T_1 \Delta_0 T_2^{-1}, \tag{5.3}$$

where T_1, T_2 are invertible over $\mathbb{Z}$ integer matrices and Δ_0 is the $\nu \times (\nu - r)$ matrix of the form

$$\Delta_0 = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{\nu-r} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Here $\sigma_1, \dots, \sigma_{\nu-r}$ are positive integers such that σ_i divides σ_{i+1}. These integers are called the *invariant factors* of the matrix S.

Since the integer $\nu \times \nu$ matrix T_1 is invertible, the columns $[T_1]^1, \dots, [T_1]^{\nu}$ form a basis of the group $\mathbb{Z}^{\nu}$ (and the vector space $\mathbb{R}^{\nu}$). Moreover, $\sigma_1[T_1]^1, \dots, \sigma_{\nu-r}[T_1]^{\nu-r}$ is a basis of the subgroup $G^{\perp}$. Indeed, from Eq. (5.3) we have

$$S = (\sigma_1[T_1]^1 \ \dots \ \sigma_{\nu-r}[T_1]^{\nu-r})T_2^{-1}.$$

Hence the columns of S are linear combinations of the columns $\sigma_1[T_1]^1, \dots, \sigma_{\nu-r}[T_1]^{\nu-r}$. Thus $\sigma_1[T_1]^1, \dots, \sigma_{\nu-r}[T_1]^{\nu-r}$ also is a generating system of $G^{\perp}$. Evidently, these columns are linearly independent over $\mathbb{Z}$, and $\sigma_1[T_1]^1, \dots, \sigma_{\nu-r}[T_1]^{\nu-r}$ is the required basis of $G^{\perp}$.

Now we can pass to a description of the subgroup $\bar{G} = (G^{\perp})^{\perp}$. Since $[T]_i[T]^j = \delta_{ij}$, where δ_{ij} is the Kronecker delta, the rows $[T_1^{-1}]_1, \dots, [T_1^{-1}]_{\nu}$ of the matrix T_1^{-1} form a basis of the vector space $\mathbb{R}^{\nu}$ which is dual with the basis $[T_1]^1, \dots, [T_1]^{\nu}$.

Let N be the discrete group generated by the vectors $\frac{1}{\sigma_1}[T_1^{-1}]_1, \dots, \frac{1}{\sigma_{\nu-r}}[T_1^{-1}]_{\nu-r}$:

$$N = \left\{ \frac{n_1}{\sigma_1}[T_1^{-1}]_1 + \cdots + \frac{n_{\nu-r}}{\sigma_{\nu-r}}[T_1^{-1}]_{\nu-r} \right\}_{(n_1, \dots, n_{\nu-r}) \in \mathbb{Z}^{\nu-r}},$$

M the r-dimensional space which is spanned by the rows $[T_1^{-1}]_{v-r+1}, \dots, [T_1^{-1}]_v$:

$$M = \{x_1[T_1^{-1}]_{v-r+1} + \cdots + x_r[T_1^{-1}]_v\}_{(x_1,\dots,x_r)\in\mathbb{R}^r}.$$

By virtue of invertibility of T_1, we have $N \cap M = 0$. Let us form the group $N \dotplus M$ and prove that

$$\bar{G} = N \dotplus M. \tag{5.4}$$

Recall

$$\bar{G} = (G^{\perp})^{\perp} = \left\{ X \in \mathbb{R}^v \;\middle|\; \sum_{j=1}^{v} x_j h_j \in \mathbb{Z},\ \ h \in G^{\perp} \right\},$$

where

$$X = (x_1, \dots, x_v) \in \mathbb{R}^v, \quad h = \begin{pmatrix} h_1 \\ \vdots \\ h_v \end{pmatrix} \in G^{\perp}.$$

Therefore if $X \in \bar{G}$, then, taking into account that $\sigma_1[T_1]^1, \dots, \sigma_{v-r}[T_1]^{v-r}$ belong to $G^{\perp}$, we have

$$XT_1 = \left(\frac{n_1}{\sigma_1}, \dots, \frac{n_{v-r}}{\sigma_{v-r}}, \alpha_{v-r+1}, \dots, \alpha_v \right),$$

where $n_1, \dots, n_{v-r}$ are integers and $\alpha_{v-r+1}, \dots, \alpha_v$ are real numbers. This means that

$$X = \left(\frac{n_1}{\sigma_1}, \dots, \frac{n_{v-r}}{\sigma_{v-r}}, \alpha_{v-r+1}, \dots, \alpha_v \right) T_1^{-1}, \tag{5.5}$$

i.e. $X \in N \dotplus M$.

Conversely, if X has form (5.5), then

$$\sigma_j X [T_1]^j = n_j \in \mathbb{Z}, \quad j = 1, \dots, v - r.$$

Since $\{\sigma_j[T_1]^j\}_{j=1}^{v-r}$ is a basis of $G^{\perp}$, $Xh \in \mathbb{Z}$ for an arbitrary $h \in G^{\perp}$, that is $X \in \bar{G}$. Formula (5.4) is proved.

Let $N_0 = N \cap \mathbb{Z}^v$ and $M_0 = M \cap \mathbb{Z}^v$. These sets are subgroups of $\mathbb{Z}^v$. Since the columns $[T_1^{-1}]_1, \dots, [T_1^{-1}]_v$ belong to $N_0 \dotplus M_0$ and generate $\mathbb{Z}^v$, we have $\mathbb{Z}^v = N_0 \dotplus M_0$. Then $\bar{G}/\mathbb{Z}^v$ is identified with the group $(N/N_0) \times (M/M_0)$. Thus for $\mathbb{F} = \bar{G}/\mathbb{Z}^v$ we have

$$\mathbb{F} = \mathbb{K} \times \mathbb{F}_0.$$

Here $\mathbb{K} = N/N_0$ is a finite group that is the direct product of $v - r$ cyclic groups $\mathbb{K}_1, \dots, \mathbb{K}_{v-r}$ of orders $\sigma_1, \dots, \sigma_{v-r}$, respectively, and $\mathbb{F}_0 = M/M_0$ is isomorphic to the torus $\mathbb{T}^r$ (see, e.g., [13, Chapter VII, Section 1, Item 5]).

Let us now find the invariant factors $\sigma_1, \dots, \sigma_{v-r}$. It is known that a compact abelian group H is monothetic iff its dual group $\hat{H}$ is topologically isomorphic to a subgroup of the group $\mathbb{T}_d$ ($\mathbb{T}_d$ is the group $\mathbb{T}$ endowed with the discrete topology) (see [26, Theorem 20]). Let H_0 be a closed subgroup of H. Since $\widehat{(H/H_0)}$ is isomorphic to the subgroup $H_0^{\perp}$ of the group $\hat{H}$ ([26, Theorem 20]), we obtain that if

H is monothetic, then the factor group H/H_0 is also monothetic. Hence $\mathbb{K} \cong \mathbb{F}/\mathbb{F}_0$ is monothetic. But this means that the finite group $\mathbb{K}$ is cyclic.

Thus $\mathbb{K}$ is a cyclic group that is the direct product of $v-r$ cyclic groups of orders $\sigma_1, \dots, \sigma_{v-r}$, where σ_i divides σ_{i+1}:

$$\mathbb{K} = \mathbb{K}_1 \times \cdots \times \mathbb{K}_{v-r}.$$

Let us prove that this is possible only if

$$\sigma_1 = \cdots = \sigma_{v-r-1} = 1, \quad \sigma_{v-r} = \sigma,$$

where σ is the order of the group $\mathbb{K}$. Suppose that there exists $\sigma_i \neq 1$ for some index i, $1 \leqslant i \leqslant v-r-1$. Let $\sigma_i = p_1^{m_1} \cdots p_l^{m_l}$ be the canonical decomposition of the integer σ_i into the product of prime numbers. Here there is j, $1 \leqslant j \leqslant l$, such that $m_j \neq 0$. Since σ_i divides σ_{i+1}, we have $\sigma_{i+1} = p_1^{k_1} \cdots p_l^{k_l}$, where $m_1 \leqslant k_1, \dots, m_l \leqslant k_l$. Then in the subgroup $\mathbb{K}_i$ ($\mathbb{K}_{i+1}$), by the first Silow's theorem (see, e.g., [21, Chapter 13, Section 54]), there exists a cyclic subgroup of order $p_j^{m_j}$ ($p_j^{k_j}$). Hence in the group $\mathbb{K}$ there exists a subgroup which is the direct product of these cyclic groups. Since $\mathbb{K}$ is a finite cyclic group, this subgroup is a decomposable cyclic group of order $p_j^{m_j+k_j}$. But this is impossible (see [21, Chapter V, Section 17]). Thus $\mathbb{K}$ is isomorphic to the cyclic group $\mathbb{Z}_\sigma$ of order σ.

It remains to find σ. As is well known, the invariant factors of a matrix S are calculated via the greatest common divisors δ_k of minors of order k of this matrix: $\delta_k = \sigma_1\sigma_2 \cdots \sigma_k$ (see [33, Chapter 12, Section 85]). Hence σ is the greatest common divisors of minors of order $v-r$ for the matrix S.

We note that although the matrix S (i.e. a basis of $G^\perp$) is not uniquely found, the integer σ does not depend on a choice of S. Thus r and σ are invariants of the given system $\Theta_0, \dots, \Theta_v$. It is easily seen that for $r=0$ (i.e. for the case when all Θ_j are rational numbers) σ coincides with the least common multiple of the denominators of Θ_j.

Let Q_j be the $(v-r+j)$th row of the matrix T_1^{-1}, $j = 0, \dots, r$. For a vector $Q = (q_1, \dots, q_v)$ we denote

$$e^{2\pi i Q} = (e^{2\pi i q_1}, \dots, e^{2\pi i q_v}).$$

Now from formula (5.4) we see that a vector $(\varphi_1, \dots, \varphi_v)$ belongs to $\bar{G}$ iff it can be represented as follows:

$$(\varphi_1, \dots, \varphi_v) = n_1[T_1^{-1}]_1 + \cdots + n_{v-r-1}[T_1^{-1}]_{v-r-1} + \frac{n}{\sigma}Q_0 + x_1Q_1 + \cdots + x_rQ_r,$$

where $n_1, \dots, n_r, n$ are arbitrary integers and $x_1, \dots, x_r$ are arbitrary real numbers. Hence a point $\tau = (\tau_1, \dots, \tau_v) \in \mathbb{T}^v$ belongs to $\mathbb{F}$ iff τ is represented in the form

$$\tau = e^{\frac{2\pi i n Q_0}{\sigma}} e^{2\pi i x_1 Q_1} \cdots e^{2\pi i x_r Q_r}.$$

Denote $\xi_n = e^{\frac{2\pi i n}{\sigma}}$, $n = 0, \dots, \sigma-1$, and put $t_k = e^{2\pi i x_k}$, $k = 1, \dots, r$. Then the map $(n, t_1, \dots, t_r) \mapsto \tau = \xi_n^{Q_0} t_1^{Q_1} \cdots t_r^{Q_r}$ is an isomorphism of the groups $\mathbb{Z}_\sigma \times \mathbb{T}^r$ and $\mathbb{F}$. The theorem is proved. $\square$

It is not difficult to write a program to compute $\mathbb{F}$ for input A.

Let us now show that the closure of the semigroup $\{\xi^k\}_{k\geqslant 0}$ also coincides with $\mathbb{F}$.

Theorem 5.1. *For any $\tau\in\mathbb{F}$ there exists the sequence Λ_τ of indices $k_1, k_2, \ldots, k_j < k_{j+1}$, such that*

$$\lim_{k\to\infty} (e^{2\pi ik\Theta_1}, \ldots, e^{2\pi ik\Theta_v}) = \tau, \quad k\in\Lambda_\tau.$$

Proof. Let $\tau = (e^{2\pi i\varphi_1}, \ldots, e^{2\pi i\varphi_v})\in\mathbb{F}$. Then $(\varphi_1, \ldots, \varphi_v)\in\bar{G}$, i.e. for any $\varepsilon>0$ there exist integers $m_1, \ldots, m_v, m$ such that

$$|m\Theta_j - \varphi_j + m_j| < \varepsilon$$

for all $j = 1, \ldots, v$. Let us show that we can always choose m greater than any given $N\in\mathbb{N}$. Indeed, for given ε first we determine $\tilde{m}_1, \ldots, \tilde{m}_v, \tilde{m}$ such that

$$|\tilde{m}\Theta_j - \varphi_j + \tilde{m}_j| < \frac{\varepsilon}{2}, \quad j = 1, \ldots, v.$$

For $\tilde{m}\geqslant N$, there is nothing to prove. Let $\tilde{m}<N$. Since $(0, \ldots, 0)\in\bar{G}$, we can choose integers $l_1, \ldots, l_v, l$ such that $l>0$ and

$$|l_j\Theta_j + l| < \frac{\varepsilon}{2(N-\tilde{m})}, \quad j = 1, \ldots, v.$$

Hence if we put $m = \tilde{m} + l(N-\tilde{m})$, $m_j = \tilde{m}_j + l(N-\tilde{m})$, we obtain the inequalities $m>N$ and

$$|m\Theta_j - \varphi_j + m_j| < \varepsilon.$$

This means that there exists a sequence Λ_τ of increasing numbers $k_1, k_2, \ldots$ such that

$$\lim_{k\to\infty} (e^{2\pi ik\Theta_1}, \ldots, e^{2\pi ik\Theta_v}) = \tau.$$

The theorem is proved. $\square$

Thus we obtain the description of the parameter group $\mathbb{F}$ and now we can study the asymptotic of the minimal solutions $V_k(z)$.

6. Asymptotic behavior of denominators of Padé approximants for rational functions

In this section we prove the main Theorem 2.2 while for rational functions only. We will obtain all limit points of the suitable normalized sequence $\{Q_n(z)\}$, where $Q_n(z)$ is the denominators of the Padé approximant of type $(n, \lambda-1)$ for the rational function $r(z) = \frac{N(z)}{D(z)}$. As we have seen in Section 4, this problem is reduced to the study of the asymptotic of the minimal solutions $V_k(z)$ of the Bezout equation because $Q_n(z) = V_{n+\lambda}(z)$. By Theorem 4.2, the polynomials $V_k(z)$ are expressed via their leading coefficients v_k. For v_k we have the formula

$$v_k = p_1(k)z_1^k + \cdots + p_\ell(k)z_\ell^k.$$

First we obtain the asymptotic for v_k. Fix any point $\tau = (\tau_1, \ldots, \tau_\nu)$ belonging to the group $\mathbb{F}$. Let Λ_τ be any sequence of numbers $k_1, k_2, \ldots,\ k_j < k_{j+1}$ such that

$$\lim_{k\to\infty} (e^{2\pi ik\Theta_1}, \ldots, e^{2\pi ik\Theta_\nu}) = \tau, \quad k \in \Lambda_\tau \subset \mathbb{N}.$$

Then it is evident that for $k\to\infty$, $k\in\Lambda_\tau$, the sequence v_k has the following asymptotic:

$$v_k = \rho^k k^{s-1}[C_1\tau_1 + \cdots + C_\nu\tau_\nu + o(1)]. \tag{6.1}$$

Recall that $\rho = |z_1| = \cdots = |z_\mu|$, s is the multiplicity $z_1, \ldots, z_\nu$, and C_j is the leading coefficient of the polynomial $p_j(z)$ (see formula (4.14)). Moreover, for all fixed $j \geqslant -k$ we have

$$v_{k+j} = \rho^k k^{s-1}[C_1 z_1^j \tau_1 + \cdots + C_\nu z_\nu^j \tau_\nu + o(1)], \quad k\to\infty,\ k\in\Lambda_\tau. \tag{6.2}$$

Denote

$$S_j(\tau) = C_1 z_1^j \tau_1 + \cdots + C_\nu z_\nu^j \tau_\nu, \quad \tau\in\mathbb{F}. \tag{6.3}$$

If $S_0(\tau)\neq 0$, then there exists

$$\lim_{k\to\infty} \frac{v_{k+j}}{v_k} = \frac{S_j(\tau)}{S_0(\tau)}, \quad k\in\Lambda_\tau.$$

We note that if the roots of $D(z)$ have distinct moduli, then we can take $\Lambda_\tau = \mathbb{N}$ and the above equality is the elementary case of the Pouncaré theorem (see [16, Chapter V, Section 5.1]).

If $S_0(\tau)\neq 0$, then the point $\tau = (\tau_1, \ldots, \tau_\nu)\in\mathbb{F}$ is called *regular* with respect to the system of the poles $z_1, \ldots, z_\nu$.

We study singular points. Since the Vandermonde matrix is invertible, from formula (6.3) and the inequalities $C_j\neq 0$ we immediately obtain

Proposition 6.1. *For any k in the sequence $S_k(\tau), S_{k+1}(\tau), \ldots, S_{k+\nu-1}(\tau)$ there exists at least one nonzero number.*

Definition 6.1. The nonnegative integer $\delta_+(\tau)$ $(\delta_-(\tau))$ will be called the plus-defect (minus-defect) of the point $\tau\in\mathbb{F}$ if $\delta_+(\tau)$ $(\delta_-(\tau))$ is the least nonnegative integer such that $S_{\delta_+(\tau)}(\tau)\neq 0$ $(S_{-\delta_-(\tau)-1}(\tau)\neq 0)$.

Thus if $\delta_+(\tau) = 0$ $(\delta_-(\tau) = 0)$, then $S_0\neq 0$ $(S_{-1}\neq 0)$. In the case $\delta_+(\tau)>0$ $(\delta_-(\tau)>0)$ we have

$$S_0 = \cdots = S_{\delta_+(\tau)-1} = 0, \quad S_{\delta_+(\tau)}\neq 0$$

$$(S_{-1} = \cdots = S_{-\delta_-(\tau)} = 0, \quad S_{-\delta_-(\tau)-1}\neq 0).$$

It follows from Proposition 6.1 that $0\leqslant\delta_+(\tau)\leqslant\nu-1$, $1\leqslant\delta_-(\tau)\leqslant\nu$. If τ is fixed, then for brevity we will write δ_+, δ_- instead of $\delta_+(\tau)$, $\delta_-(\tau)$.

Let δ_+ be the plus-defect of the point $\tau \in \mathbb{F}$. Then from asymptotic (6.2) we obtain

$$\lim_{k\to\infty} \frac{v_{k+j}}{v_{k+\delta_+}} = \frac{S_j}{S_{\delta_+}}, \quad k \in \Lambda_\tau. \tag{6.4}$$

We are now ready to study the asymptotic of the minimal solutions.

Theorem 6.1. *Let τ be an arbitrary point of the group $\mathbb{F}$ and let Λ_τ be a sequence of indices corresponding to τ.*

Then, for all $k \in \Lambda_\tau$ sufficiently large the polynomial $V_k(z)$ can be $(\lambda - \delta_+ - 1)$-normalized and there exists

$$\lim_{k\to\infty} V_k^{(\lambda-\delta_+-1)}(z) = W(z,\tau), \quad k \in \Lambda_\tau.$$

Here $V_k^{(\lambda-\delta_+-1)}(z)$ is the $(\lambda-\delta_+-1)$-normalized polynomial $V_k(z)$, $W(z,\tau)$ is a polynomial of degree $\lambda-\delta_+-1$ which has $z=0$ as a root of multiplicity δ_-. This polynomial is computed by the formula:

$$W(z,\tau) = S_{\delta_+}^{-1}(\tau)\omega(z,\tau)(z-z_1)^{s_1-1}\dots(z-z_v)^{s_v-1}(z-z_{v+1})^{s_{v+1}}\dots(z-z_\ell)^{s_\ell},$$

$$\omega(z,\tau) = \sum_{j=1}^{v} C_j\Delta_j(z)\tau_j, \quad \Delta_j = \frac{\Delta(z)}{z-z_j}, \quad \Delta(z) = (z-z_1)\dots(z-z_v). \tag{6.5}$$

Proof. By formula (4.7), the coefficient of $z^{\lambda-\delta_+-1}$ in the polynomial $V_k(z)$ is found as follows:

$$\alpha_{k,\lambda-\delta_+-1} = \sum_{i=1}^{\delta_++1} d_{\lambda-i+1}v_{k+\delta_+-i+1}.$$

From asymptotic (6.2) it follows that for all $k \in \Lambda_\tau$ sufficiently large the coefficient $v_{k+\delta_+}$ is nonzero. Therefore,

$$\alpha_{k,\lambda-\delta_+-1} = v_{k+\delta_+}\beta_{k,\lambda-\delta_+-1},$$

where

$$\beta_{k,\lambda-\delta_+-1} = 1 + d_{\lambda-1}\frac{v_{k+\delta_+-1}}{v_{k+\delta_+}} + \dots + d_{\lambda-\delta_+}\frac{v_k}{v_{k+\delta_+}}.$$

From the definition of the defect δ_+ and formula (6.4) we have

$$\lim_{k\to\infty}\frac{v_{k+\delta_+-1}}{v_{k+\delta_+}} = \dots = \lim_{k\to\infty}\frac{v_k}{v_{k+\delta_+}} = 0, \quad k \in \Lambda_\tau.$$

Hence $\lim_{k\to\infty}\beta_{k,\lambda-\delta_+-1} = 1$, $k \in \Lambda_\tau$, and $\alpha_{k,\lambda-\delta_+-1}$ is nonzero for all $k \in \Lambda_\tau$ sufficiently large. Thus the polynomial $V_k(z)$ can be really $(\lambda-\delta_+-1)$-normalized. In order to obtain the normalized polynomial $V_k^{(\lambda-\delta_+-1)}(z)$ we use formula (4.8):

$$V_k^{(\lambda-\delta_+-1)}(z) = \beta_{k,\lambda-\delta_+-1}^{-1}\sum_{i=1}^{\lambda}\sum_{j=0}^{i-1} d_i\frac{v_{k+i-j-1}}{v_{k+\delta_+}}z^j.$$

Consequently there exists

$$\lim_{k\to\infty} V_k^{(\lambda-\delta_+-1)}(z) = S_{\delta_+}^{-1} \sum_{i=1}^{\lambda} \sum_{j=0}^{i-1} d_i S_{i-j-1} z^j, \quad k \in \Lambda_\tau.$$

This limit we will denote by $W(z,\tau)$. It is easily seen that the coefficient $W_{\lambda-j}$ of $z^{\lambda-j}$ in the polynomial $W(z,\tau)(z)$ is

$$W_{\lambda-j} = S_{\delta_+}^{-1} \sum_{i=1}^{j} d_{\lambda-i+1}\, S_{j-i}.$$

By the definition of the defect δ_+, we have

$$S_0 = \cdots = S_{\delta_+-1} = 0, \quad S_{\delta_+} \neq 0.$$

Hence $W_{\lambda-j} = 0$ for $j = 1,\dots,\delta_+$ and $W_{\lambda-\delta_+-1} = 1$. Thus the degree of $W(z,\tau)$ coincides with $\lambda-\delta_+-1$ and the leading coefficient equals 1.

Let us obtain formula (6.5) for $W(z,\tau)$. First we find $\sum_{j=0}^{i-1} S_{i-j-1} z^j$. Taking into account definition (6.3) for S_{i-j-1}, we get

$$\sum_{j=0}^{i-1} S_{i-j-1} z^j = \sum_{j=0}^{i-1} \sum_{l=1}^{v} C_l \tau_l z_l^{i-j-1} z^j = \sum_{l=1}^{v} C_l \tau_l \sum_{j=0}^{i-1} z_l^{i-j-1} z^j = \sum_{l=1}^{v} C_l \tau_l \frac{z^i - z_l^i}{z - z_l}.$$

Therefore,

$$\begin{aligned} W(z,\tau) &= S_{\delta_+}^{-1} \sum_{i=1}^{\lambda} \sum_{l=1}^{v} d_i C_l \tau_l \frac{z^i - z_l^i}{z - z_l} = S_{\delta_+}^{-1} \sum_{l=1}^{v} \frac{C_l \tau_l}{z - z_l} \sum_{i=0}^{\lambda} d_i (z^i - z_l^i) \\ &= S_{\delta_+}^{-1} \sum_{l=1}^{v} C_l \tau_l \frac{D(z)}{z - z_l}. \end{aligned}$$

Since

$$D(z) = \Delta(z)(z-z_1)^{s_1-1} \dots (z-z_v)^{s_v-1} (z-z_{v+1})^{s_{v+1}} \dots (z-z_\ell)^{s_\ell},$$

we arrive at formula (6.5).

It remains to prove that δ_- is the multiplicity of $z=0$ as a root of the polynomial $W(z,\tau)$. Obviously, the polynomials $W(z,\tau)$, $\omega(z,\tau)$ and the analytic function $\sum_{j=1}^{v} \frac{C_j \tau_j}{z - z_j}$ have $z=0$ as a root of the same multiplicity. Expanding the function $\sum_{j=1}^{v} \frac{C_j \tau_j}{z - z_j}$ into a series in power of z, we obtain

$$\sum_{j=1}^{v} \frac{C_j \tau_j}{z - z_j} = S_{-1} + S_{-2} z + S_{-3} z^2 + \cdots.$$

By the definition of δ_-, we have

$$\sum_{j=1}^{v} \frac{C_j \tau_j}{z - z_j} = S_{-\delta_- -1} z^{\delta_-} + S_{-\delta_- -2} z^{\delta_- +1} + \cdots.$$

This means that δ_- is the multiplicity of $z=0$ as a root of $W(z,\tau)$. The theorem is proved. $\square$

Thus the asymptotic of the minimal solutions $V_k(z)$ is obtained.

7. Preparation Theorem

In this section we prove the preparation theorem that allows to reduce the study of the asymptotic behavior of the denominators $Q_n(z)$ of the Padé approximants for a meromorphic function $a(z)$ to the same problem for the rational part $r(z)$ of $a(z)$.

Let $a(z)$ be a function which is meromorphic in the disk $|z|<R$ and analytic at the origin. Without loss of generality we may suppose that $R>1$. Let $r(z)$ be the sum of the principal parts of the Laurent series of $a(z)$ in neighborhoods of the poles of $a(z)$ (the rational part of $a(z)$). Obviously, $r(z)$ is a strictly proper rational function that is analytic at $z=0$. Let us represent $a(z)$ as follows:

$$a(z) = b(z) + r(z),$$

where the function $b(z)$ is analytic in $|z|<R$. Expand these functions into the Taylor series in power of z:

$$\sum_{j=0}^{\infty} a_j z^j = \sum_{j=0}^{\infty} r_j z^j + \sum_{j=0}^{\infty} b_j z^j.$$

In order to construct denominator of the Padé approximant of type (n,m) we need the sequence $a_{n-m+1}^{n+m} = \{a_{n-m+1}, a_{n-m+2}, \dots, a_{n+m}\}$. We will consider this sequence as a perturbation of the sequence r_{n-m+1}^{n+m} by the sequence b_{n-m+1}^{n+m}. Since the last sequence is infinitesimal as $n\to\infty$, it can be really regarded as a small perturbation. We consider the smallness with respect to the following norms.

For matrices in $\mathbb{C}^{l\times k}$ we will use the norm

$$||A|| = \max_{1\leqslant j\leqslant l} \sum_{i=1}^{k} |A_{ij}|,$$

and for a sequence $c_M^N = \{c_M, c_{M+1}, \dots, c_N\}$ we introduce

$$||c_M^N|| = \sum_{i=M}^{N} |c_i|.$$

We will use the same norm for the generating polynomial $c_M^N(z) = c_M z^M + c_{M+1} z^{M+1} + \dots + c_N z^N$ of this sequence. It is easily seen that for the Toeplitz matrix $T_k(c_M^N)$ $(M\leqslant k\leqslant N)$ holds

$$||T_k(c_M^N)|| \leqslant ||T_M(c_M^N)|| = ||c_M^N||.$$

By results of Section 3, to study of the asymptotic behavior of the denominators $Q_n(z)$ of the Padé approximants of type $(n,\lambda-1)$, we must investigate the behavior of the indices and essential polynomials for the sequence $a_{n-\lambda+2}^{n+\lambda-1}$. The indices and essential polynomials of the nonperturbed sequence $r_{n-\lambda+2}^{n+\lambda-1}$ we have already found (see Section 4). Now we prove that the indices of $a_{n-\lambda+2}^{n+\lambda-1}$ are equal to the indices of $r_{n-\lambda+2}^{n+\lambda-1}$, and the essential polynomials of these sequences are infinitesimally close.

To prove this, we need the lemma on the continuity of projectors on the kernels of Fredholm operators. We reformulate this well-known result (see, e.g., [17, Theorem 11.3]) in the form that is suitable for us.

Lemma 7.1. *Let A be a right invertible operator from a Banach space E_1 into a Banach space E_2 such that* $\ker A$ *is finite-dimensional. Let $A^\dagger$ be any right inverse of A and $P_A = I - A^\dagger A$ is the projector on* $\ker A$. *Then for any operator B such that $||A - B|| < \frac{1}{2||A^\dagger||}$ it is fulfilled*:

1. *B is right invertible and $B^\dagger = C^{-1}A^\dagger$ is the right inverse of B. Here* $C = I - A^\dagger(A - B)$, $C^{-1} = \sum_{i=0}^{\infty}(A^\dagger(A-B))^i$.
2. *For the projector $P_B = I - B^\dagger B = C^{-1}P_A C$ on* $\ker B$ *the following inequality holds*

$$||P_A - P_B|| < 4||P_A||\ ||A^\dagger||\ ||A - B||.$$

To apply this lemma we will need an estimate of the norm $||P_A||$ for $A = T_{n+1}(r_{n-\lambda+2}^{n+\lambda-1})$. Unfortunately, in our case the trivial estimate $||P_A|| \leqslant 1 + ||A^\dagger||||A||$ is useless. Therefore we will choose the projector P_A (i.e. a right inverse of A) by a special way.

In view of further applications to the other intermediate rows, in the following auxiliary propositions and in Preparation Theorem we will consider arbitrary sequences c_{n-m+1}^{n+m} with the stable indices, where m cannot be equal to $\lambda - 1$.

Proposition 7.1. *Let c_{n-m+1}^{n+m} be any sequence with the stable indices n, $n+1$. The first essential polynomial $R_1(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$ of this sequence has the nonzero coefficient α_d if and only if the matrix $T_{n+1,d+1}(c_{n-m+1}^{n+m})$ obtained from the $m \times (m+1)$ matrix $T_{n+1}(c_{n-m+1}^{n+m})$ by deleting of the $(d+1)$th column is invertible.*

Suppose that this condition is fulfilled. Let $R_1(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$ be the first essential d-normalized polynomial. Then the matrix

$$\mathscr{P}_{n+1}(c_{n-m+1}^{n+m}) = \begin{pmatrix} 0 & \dots & \alpha_0 & \dots & 0 \\ 0 & \dots & \alpha_1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & \alpha_m & \dots & 0 \end{pmatrix} \tag{7.1}$$

is the matrix of a projector on $\ker T_{n+1}(c_{n-m+1}^{n+m})$. *Here the nonzero column has the number $d+1$. Moreover,*

$$\mathscr{P}_{n+1}(c_{n-m+1}^{n+m}) = I_{m+1} - T_{n+1}^{\dagger}(c_{n-m+1}^{n+m})T_{n+1}(c_{n-m+1}^{n+m}),$$

where $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$ *is the right inverse of* $T_{n+1}(c^{n+m}_{n-m+1})$ *obtained from the matrix* $T^{-1}_{n+1,d+1}(c^{n+m}_{n-m+1})$ *by adding the zero row in the position with the number* $d+1$.

Proof. Suppose the matrix $T_{n+1,d+1}(c^{n+m}_{n-m+1})$ is invertible but the coefficient α_d of the polynomial $R_1(z)$ is zero. Since the vector R_1 consisting of the coefficients of the polynomial $R_1(z)$ belongs to $\ker T_{n+1}(c^{n+m}_{n-m+1})$, we have $R_1(z)\equiv 0$, which is impossible. Hence $\alpha_d \neq 0$.

Conversely, if $\alpha_d \neq 0$ and the matrix $T_{n+1,d+1}(c^{n+m}_{n-m+1})$ is singular, then the space $\ker T_{n+1}(c^{n+m}_{n-m+1})$ contains at least two linearly independent vectors, which is also impossible because this space is one-dimensional in virtue of the stability of the indices.

Let $T_{n+1,d+1}(c^{n+m}_{n-m+1})$ be an invertible matrix and $\alpha_d = 1$. The direct calculations show that $\mathscr{P}_{n+1}(c^{n+m}_{n-m+1})$ is a projector. It is also obvious that

$$\operatorname{Im} \mathscr{P}_{n+1}(c^{n+m}_{n-m+1}) = \operatorname{span}\{R_1\} = \ker T_{n+1}(c^{n+m}_{n-m+1}).$$

Hence $\mathscr{P}_{n+1}(c^{n+m}_{n-m+1})$ is a projector on $\ker T_{n+1}(c^{n+m}_{n-m+1})$. Since the indices of c^{n+m}_{n-m+1} are $n,\ n+1$, the $m\times(m+1)$ matrix $T_{n+1}(c^{n+m}_{n-m+1})$ is right invertible. Suppose the projector $\mathscr{P}_{n+1}(c^{n+m}_{n-m+1})$ is generated by some right inverse $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$, i.e.,

$$\mathscr{P}_{n+1}(c^{n+m}_{n-m+1}) = I_{m+1} - T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})T_{n+1}(c^{n+m}_{n-m+1}).$$

Then from the equality

$$\mathscr{P}_{n+1}(c^{n+m}_{n-m+1})T^{\dagger}_{n+1}(c^{n+m}_{n-m+1}) = 0$$

it follows that $[T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})]_{d+1}$ is the zero row.

Denote by $T^{\dagger}_{n+1,d+1}(c^{n+m}_{n-m+1})$ the matrix obtained from $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$ by deleting of this zero row. Then from the equation

$$T_{n+1}(c^{n+m}_{n-m+1})T^{\dagger}_{n+1}(c^{n+m}_{n-m+1}) = I_m$$

we get

$$T_{n+1,d+1}(c^{n+m}_{n-m+1})T^{\dagger}_{n+1,d+1}(c^{n+m}_{n-m+1}) = I_m,$$

i.e. $T^{\dagger}_{n+1,d+1}(c^{n+m}_{n-m+1}) = T^{-1}_{n+1,d+1}(c^{n+m}_{n-m+1})$. This completes the proof. $\square$

Corollary 7.1.

$$||\mathscr{P}_{n+1}(c^{n+m}_{n-m+1})|| = ||R_1(z)||, \tag{7.2}$$

where $R_1(z)$ *is the d-normalized first essential polynomial of the sequence* c^{n+m}_{n-m+1}.

To apply Lemma 7.1, we must also obtain an estimate for the norm of the constructed matrix $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$. First we find an explicit formula for $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$ in terms of the essential polynomials. Exactly here in our approach the second essential polynomial is appeared.

Proposition 7.2. *Let c_{n-m+1}^{n+m} be any sequence with the stable indices $n, n+1$, $R_1(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$, $R_2(z) = \beta_0 + \beta_1 z + \cdots + \beta_{m+1} z^{m+1}$ its essential polynomials, and $\alpha_d = 1$.*

The right inverse $T_{n+1}^{\dagger}(c_{n-m+1}^{n+m})$ from Proposition 7.1 *has the following form*

$$T_{n+1}^{\dagger}(c_{n-m+1}^{n+m}) = G - K,$$

where

$$G = \frac{1}{\sigma_0}\left[\begin{pmatrix} \alpha_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \alpha_m & \dots & \alpha_0 \end{pmatrix}\begin{pmatrix} \beta_m & \dots & \beta_1 \\ \beta_{m+1} & \dots & \beta_2 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \beta_{m+1} \end{pmatrix} - \begin{pmatrix} \beta_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \beta_m & \dots & \beta_0 \end{pmatrix}\begin{pmatrix} \alpha_m & \dots & \alpha_1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_m \\ 0 & \dots & 0 \end{pmatrix}\right], \tag{7.3}$$

$$K = \begin{pmatrix} g_{d+1,1}R_1 & g_{d+1,2}R_1 & \dots & g_{d+1,m}R_1 \end{pmatrix},$$

$$[G]_{d+1} = \begin{pmatrix} g_{d+1,1} & g_{d+1,2} & \dots & g_{d+1,m} \end{pmatrix}, \quad R_1 = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$$

and $\sigma_0 = \sigma\{z^{-n}\beta_{m+1}R_1(z) - z^{-n-1}\alpha_m R_2(z)\}$ is the test number for $R_1(z)$, $R_2(z)$.

Proof. Since the indices c_{n-m+1}^{n+m} are stable, the matrix $T_{n+1}(c_{n-m+1}^{n+m})$ is right invertible. First we construct the right inverse G of $T_{n+1}(c_{n-m+1}^{n+m})$ by the formula from the work [3] (see also [7, Corollary 7.3]). Define the generating polynomial for $G = ||g_{ij}||_{\substack{i=0,\dots,m-1 \\ j=0,1,\dots,m}}$ as follows:

$$G(t,s) = \sum_{i=0}^{m-1}\sum_{j=0}^{m} g_{ij}t^i s^{-j}.$$

Then $G(t,s)$ is found by the formula

$$G(t,s) = \mathscr{P}_t(0, m-1)\mathscr{P}_s(-m, 0)s\mathscr{B}(t,s),$$

where

$$\mathscr{B}(t,s) = \frac{1}{\sigma_0}s^{-m}\frac{R_1(t)R_2(s) - R_1(s)R_2(t)}{1 - ts^{-1}}$$

is the generating function for the matrix

$$B = \frac{1}{\sigma_0}\left[\begin{pmatrix} \alpha_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \alpha_m & \dots & \alpha_0 \end{pmatrix}\begin{pmatrix} \beta_{m+1} & \dots & \beta_1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \beta_{m+1} \end{pmatrix} - \begin{pmatrix} \beta_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \beta_m & \dots & \beta_0 \end{pmatrix}\begin{pmatrix} \alpha_{m+1} & \dots & \alpha_1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_{m+1} \end{pmatrix}\right].$$

Here, for convenience, we put $\alpha_{m+1} = 0$, and $\mathscr{P}(\alpha, \beta)$ is the projector acting by the formula

$$\mathscr{P}(\alpha, \beta) \sum_{i=-m}^{n} r_i t^i = \sum_{i=\alpha}^{\beta} r_i t^i.$$

The index t means that the operator acts on the variable t.

It is easily seen that the matrix G is obtained from B by deleting the first column. Hence for G we get formula (7.3).

The matrix $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1}) = G - K$ is also a right inverse of the matrix $T_{n+1}(c^{n+m}_{n-m+1})$ because $R_1 \in \ker T_{n+1}(c^{n+m}_{n-m+1})$ and hence $T_{n+1}(c^{n+m}_{n-m+1})K = 0$. Moreover, since $\alpha_d = 1$, we have $[G]_{d+1} = [K]_{d+1}$. This means that the $(d+1)$th row of $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$ is zero, i.e. $T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})$ is obtained from $T^{-1}_{n+1,d+1}(c^{n+m}_{n-m+1})$ by adding the zero row in the $(d+1)$th position. $\square$

Corollary 7.2.

$$||T^{\dagger}_{n+1}(c^{n+m}_{n-m+1})|| \leqslant \frac{2}{|\sigma_0|}||R_1(z)||\ ||R_2(z)||(1 + ||R_1(z)||\,).$$

The proof follows immediately from the estimates

$$||G|| \leqslant \frac{2}{|\sigma_0|}||R_1(z)||\ ||R_2(z)||,$$

$$||K|| = \max_{1 \leqslant i \leqslant m} |g_{d+1,i}|\ ||R_1(z)|| \leqslant \frac{2}{|\sigma_0|}||R_1(z)||^2||R_2(z)||.$$

Now we can prove the basic result of the section.

Theorem 7.1 (Preparation Theorem). *Let $a(z)$ be a meromorphic function and $r(z)$ its rational part. Suppose for some fixed m there exists a sequence Λ of numbers $n_1, n_2, \dots, n_k, \dots,\ n_k < n_{k+1}$, such that the following conditions are fulfilled*:

1. *For all $n \in \Lambda$ the indices of the sequences r^{n+m}_{n-m+1} are stable, i.e. are equal to $n, n+1$.*
2. *For all $n \in \Lambda$ there is an integer d, $0 \leqslant d \leqslant m$, such that there exist the d-normalized first essential polynomials $R^{(n,m)}_{1,r}(z)$ of these sequences.*

3. $\|R_{1,r}^{(n,m)}(z)\|$ *are uniformly bounded by* $n \in \Lambda$.
4. *For all* $n \in \Lambda$ *there exist the second essential polynomials* $R_{2,r}^{(n,m)}(z)$ *of the sequences such that* $\|R_{2,r}^{(n,m)}(z)\|/|\sigma_0^{(n,m)}|$ *are uniformly bounded by* $n \in \Lambda$.

Then for all $n \in \Lambda$ *sufficiently large we have*

1. *The indices of the sequences* a_{n-m+1}^{n+m} *are stable.*
2. *There exist the* d*-normalized first essential polynomials* $R_{1,a}^{(n,m)}(z)$ *of these sequences.*
3. $\|R_{1,a}^{(n,m)}(z) - R_{1,r}^{(n,m)}(z)\| \to 0$ *for* $n \to \infty$, $n \in \Lambda$.

Proof. For the sequence r_{n-m+1}^{n+m} all conditions of Proposition 7.1 are fulfilled. Therefore for all $n \in \Lambda$ the matrix $T_{n+1,d+1}(r_{n-m+1}^{n+m})$ is invertible. Moreover, by virtue of Corollary 7.2 and Condition 3–4, we have

$$\|T_{n+1,d+1}^{-1}(r_{n-m+1}^{n+m})\| \leqslant c_1,$$

where the constant c_1 does not depend on $n \in \Lambda$. Let us now consider the matrix $T_{n+1,d+1}(a_{n-m+1}^{n+m})$. The following estimate

$$\begin{aligned} &\|T_{n+1,d+1}(a_{n-m+1}^{n+m}) - T_{n+1,d+1}(r_{n-m+1}^{n+m})\| \\ &\quad = \|T_{n+1,d+1}(b_{n-m+1}^{n+m})\| \leqslant \|T_{n+1}(b_{n-m+1}^{n+m})\| \leqslant \|b_{n-m+1}^{n+m}\| = \sum_{j=n-m+1}^{n+m} |b_j| \end{aligned}$$

holds. Since the series $\sum_{j=0}^{\infty} b_j z^j$ is absolutely convergent on $|z| = 1$ (recall that we assume $R > 1$), for any ε there exists a number n_0 such that

$$\|T_{n+1,d+1}(a_{n-m+1}^{n+m}) - T_{n+1,d+1}(r_{n-m+1}^{n+m})\| < \varepsilon$$

for $n > n_0$, $n \in \Lambda$. If $\varepsilon < 1/c_1$, then for all $n \in \Lambda$ sufficiently large we obtain

$$\|T_{n+1,d+1}(a_{n-m+1}^{n+m}) - T_{n+1,d+1}(r_{n-m+1}^{n+m})\| \leqslant \frac{1}{\|T_{n+1,d+1}^{-1}(r_{n-m+1}^{n+m})\|},$$

i.e. the matrix $T_{n+1,d+1}(a_{n-m+1}^{n+m})$ is also invertible. Hence for all $n \in \Lambda$ sufficiently large rank $T_{n+1}(a_{n-m+1}^{n+m}) = m$ and the kernel of this matrix is one-dimensional. Thus the index μ_1 of the sequence a_{n-m+1}^{n+m} equals n for all $n \in \Lambda$ sufficiently large. Then $\mu_2 = n + 1$. Moreover, by Proposition 7.1, the first essential polynomials $R_{1,a}^{(n,m)}(z)$ of the sequences a_{n-m+1}^{n+m} can be d-normalized.

It remains to prove Statement 3. Using the d-normalized first essential polynomials $R_{1,a}^{(n,m)}(z)$, $R_{1,r}^{(n,m)}(z)$ of the sequences a_{n-m+1}^{n+m}, r_{n-m+1}^{n+m}, respectively, we form according to Proposition 7.1 the projectors $\mathscr{P}_{n+1}(a_{n-m+1}^{n+m})$, $\mathscr{P}_{n+1}(r_{n-m+1}^{n+m})$ on the kernels of the matrices $T_{n+1}(a_{n-m+1}^{n+m})$, $T_{n+1}(r_{n-m+1}^{n+m})$. Obviously,

$$||R_{1,a}^{(n,m)}(z) - R_{1,r}^{(n,m)}(z)|| = ||\mathscr{P}_{n+1}(a_{n-m+1}^{n+m}) - \mathscr{P}_{n+1}(r_{n-m+1}^{n+m})||.$$

Now we apply Lemma 7.1. Put $A = T_{n+1}(r_{n-m+1}^{n+m})$, $B = T_{n+1}(a_{n-m+1}^{n+m})$ and $A^\dagger = T_{n+1}^\dagger(r_{n-m+1}^{n+m})$. Then $P_A = \mathscr{P}_{n+1}(r_{n-m+1}^{n+m})$ and for all $n \in \Lambda$ sufficiently large the condition $||A - B|| = ||T_{n+1}(b_{n-m+1}^{n+m})|| < \frac{1}{2||A^\dagger||}$ is fulfilled. Let us prove that the projector P_B constructed in Lemma 7.1 coincides with $\mathscr{P}_{n+1}(a_{n-m+1}^{n+m})$. To do this, we must define more exactly the structure of the matrices C and C^{-1} from this lemma.

Since

$$C = I_{m+1} - T_{n+1}^\dagger(r_{n-m+1}^{n+m})T_{n+1}(b_{n-m+1}^{n+m}),$$

and $[T_{n+1}^\dagger(r_{n-m+1}^{n+m})]_{d+1}$ is the zero row (see Proposition 7.1), we obtain that the $(d+1)$th row of C has the form

$$[C]_{d+1} = (0\ldots0 \overset{d+1}{1} \ 0\ldots0).$$

It is clear that the same form has the $(d+1)$th row of the matrix

$$C^{-1} = I_{m+1} + \sum_{j=1}^{\infty} (T_{n+1}^\dagger(r_{n-m+1}^{n+m})T_{n+1}(b_{n-m+1}^{n+m}))^j.$$

Then

$$P_B = C^{-1}\mathscr{P}_{n+1}(r_{n-m+1}^{n+m})C = \begin{pmatrix} 0 & \ldots & * & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & 1 & \ldots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \ldots & * & \ldots & 0 \end{pmatrix}.$$

Here only the $(d+1)$th column is nonzero and the element 1 is situated in the position $(d+1, d+1)$. Since P_B is the projector on $\ker T_{n+1}(a_{n-m+1}^{n+m})$, this column belongs to $\ker T_{n+1}(a_{n-m+1}^{n+m})$. Hence it coincides with the d-normalized essential polynomial $R_{1,a}^{(n,m)}(z)$. This means that $P_B = \mathscr{P}_{n+1}(a_{n-m+1}^{n+m})$.

Apply Lemma 7.1. Taking into account Corollaries 7.1 and 7.2, we get

$$||\mathscr{P}_{n+1}(a_{n-m+1}^{n+m}) - \mathscr{P}_{n+1}(r_{n-m+1}^{n+m})|| < c_2||T_{n+1}(b_{n-m+1}^{n+m})|| \leqslant c_2 \sum_{j=n-m+1}^{n+m} |b_j|,$$

where c_2 is a constant independent of $n \in \Lambda$. Since the series $\sum_{j=0}^{\infty} |b_j|$ is convergent, we finally obtain

$$||R_{1,a}^{(n,m)}(z) - R_{1,r}^{(n,m)}(z)|| \to 0$$

as $n \to \infty$, $n \in \Lambda$. The Preparation Theorem is proved. $\square$

8. Asymptotic behavior of denominators of Padé approximants for meromorphic functions

Now we are ready to prove the main result of the work (Theorem 2.2). We will see that the asymptotic behavior of the denominators of the Padé approximants of type $(n, \lambda - 1)$ for a meromorphic function $a(z)$ and for its rational part $r(z)$ is identical. To obtain this result, we have to verify all conditions of Preparation Theorem 7.1. In addition to the hypothesis $R > 1$, without loss of generality we suppose that all poles of $a(z)$ lie inside the unit circle.

Proof of Theorem 2.2. Let us verify Conditions 1–4 of Preparation Theorem 7.1. We take the sequence Λ_τ shifted by λ (because the denominator $Q_n(z)$ for the rational fraction $r(z)$ is $V_{n+\lambda}(z)$) as Λ.

1. If $m = \lambda - 1$, then, by Theorem 4.1, the sequence r_{n-m+1}^{n+m} for the rational function $r(z)$ has the stable indices $n,\ n+1$ and the essential polynomials

$$\tilde{R}_{1,r}^{(n,m)}(z) = V_{n+\lambda}(z), \quad R_{2,r}^{(n,m)}(z) = D(z)$$

for which $\tilde{\sigma}_0^{(n,m)} = -1$.

2. By Theorem 6.1, for all sufficiently large n such that $n \in \Lambda_\tau - \lambda$ the first essential polynomial $\tilde{R}_{1,r}^{(n,m)}(z) = V_{n+\lambda}(z)$ admits the $(\lambda - \delta_+ - 1)$-normalization. Let $R_{1,r}^{(n,m)}(z)$ be the $(\lambda - \delta_+ - 1)$-normalized polynomial:

$$R_{1,r}^{(n,m)}(z) = V_{n+\lambda}^{(\lambda-\delta_+-1)}(z) = \frac{1}{v_{n+\lambda+\delta_+}\beta_{n+\lambda,\lambda-\delta_+-1}} V_{n+\lambda}(z).$$

Here $\beta_{n+\lambda,\lambda-\delta_+-1} \to 1$ as $n \to \infty$, $n \in \Lambda_\tau - \lambda$ (see Theorem 6.1).

Moreover, by this theorem, there exists

$$\lim_{n\to\infty} R_{1,r}^{(n,m)}(z) = W(z,\tau), \quad n \in \Lambda_\tau - \lambda.$$

3. Hence the norms $||R_{1,r}^{(n,m)}(z)||$ are uniformly bounded by n for $n \in \Lambda_\tau - \lambda$.

4. The test number $\sigma_0^{(n,m)}$ for the essential polynomials

$$R_{1,r}^{(n,m)}(z) = \frac{1}{v_{n+\lambda+\delta_+}\beta_{n+\lambda,\lambda-\delta_+-1}} V_{n+\lambda}(z), \quad R_{2,r}^{(n,m)}(z) = D(z)$$

is found by the formula

$$\sigma_0^{(n,m)} = -\frac{1}{v_{n+\lambda+\delta_+}\ \beta_{n+\lambda,\lambda-\delta_+-1}}.$$

Hence,

$$|\sigma_0^{(n,m)}|^{-1}||R_{2,r}^{(n,m)}(z)|| = |v_{n+\lambda+\delta_+}\beta_{n+\lambda,\lambda-\delta_+-1}|\ ||D(z)||.$$

In virtue of asymptotic (6.2), we have

$$v_{n+\lambda+\delta_+} = \rho^{n+\lambda}(n+\lambda)^{s-1}[S_{\delta_+} + o(1)].$$

Then $v_{n+\lambda+\delta_+} \to 0$ as $n \to \infty$, $n \in \Lambda_\tau - \lambda$, because $\rho = |z_1| < 1$. Since $\beta_{n+\lambda,\lambda-\delta_+-1}(z) \to 1$, the norms $|\sigma_0^{(n,m)}|^{-1} ||R_{2,r}^{(n,m)}(z)||$ are uniformly bounded by n, $n \in \Lambda_\tau - \lambda$. Thus all conditions of the Preparation Theorem are fulfilled.

Hence, by this theorem, the following statements hold:

1. The indices of the sequence a_{n-m+1}^{n+m} for the meromorphic function $a(z)$ for all sufficiently large n, $n \in \Lambda_\tau - \lambda$ are stable, i.e. are equal to n, $n+1$. This means that $\ker T_{n+1}(a_{n-m+1}^{n+m})$ is one-dimensional. Therefore, the denominator $Q_n(z)$ of the Padé approximant of type $(n, \lambda - 1)$ for $a(z)$ is unique up to multiplication by a constant and $Q_n(z)$ is the first essential polynomial $R_{1,a}^{(n,m)}(z)$ of a_{n-m+1}^{n+m}.

2. The polynomials $R_{1,a}^{(n,m)}(z)$, i.e. the denominators $Q_n(z)$, admit the $(\lambda - \delta_+ - 1)$-normalization.

Let $Q_n(z)$ be such a normalized polynomial. Then we have

3. $||R_{1,r}^{(n,m)}(z) - R_{1,a}^{(n,m)}(z)|| \to 0$, $n \to \infty$, $n \in \Lambda_\tau - \lambda$.

Taking into account Theorem 6.1, we finally obtain

$$\lim_{n \to \infty} Q_n(z) = \lim_{n \to \infty} V_{n+\lambda}^{(\lambda-\delta_+-1)}(z) = W(z, \tau), \quad n \in \Lambda_\tau - \lambda.$$

It remains only to prove that the family $W(z, \tau)$, $\tau \in \mathbb{F}$, exhausts all limit points of the suitable normalized subsequences of $\{\gamma_n Q_n(z)\}$, where γ_n is a normalizing factor. We except the trivial cases when the limit of a subsequence of $\{\gamma_n Q_n(z)\}$ equals zero or infinity. Then we can assume that a subsequence of $\{Q_n(z)\}$ is normalized in such a way that the coefficient of z^d in $\gamma_n Q_n(z)$ equals 1 for some d, $0 \leqslant d \leqslant m$, i.e. $\gamma_n Q_n(z)$ is d-normalized.

Thus suppose there exists a sequence Λ of numbers such that for all sufficiently large n, $n \in \Lambda - \lambda$, the denominators $Q_n(z)$ can be d-normalized and there exists

$$\lim_{n \to \infty} Q_n^{(d)}(z) = W_\Lambda(z), \quad n \in \Lambda - \lambda.$$

(It is clear that the inequality $d \leqslant \deg W_\Lambda(z)$ must be fulfilled.)

Let us consider the sequence $(e^{2\pi i n \Theta_1}, \dots, e^{2\pi i n \Theta_v})$ for $n \in \Lambda$. Since these points lie on the compact set $\mathbb{T}^v$ there is a subsequence $\Lambda_0 \subseteq \Lambda$ such that there exists

$$\lim_{n \to \infty} (e^{2\pi i n \Theta_1}, \dots, e^{2\pi i n \Theta_v}) = \tau, \quad n \in \Lambda_0$$

for some $\tau \in \mathbb{T}^v$. From the definition of $\mathbb{F}$ it follows that $\tau \in \mathbb{F}$ and hence $\Lambda_0 = \Lambda_\tau$. Then, by the proved part of the theorem, we have

$$\lim_{n \to \infty} Q_n(z) = W(z, \tau), \quad n \in \Lambda_\tau - \lambda.$$

Hence there exists $A = \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \frac{Q_n^{(d)}(z)}{Q_n(z)}$, $n \in \Lambda_\tau - \lambda$.

Thus $W_\Lambda(z) = AW(z, \tau)$. This completes the proof. $\square$

Thus the asymptotic of the denominators $Q_n(z)$ is completely studied.

Proposition 2.1 follows at once from continuity of roots of polynomials. Here we assume that $Q_n(z)$ has the root $z = \infty$ of corresponding multiplicity if $\deg Q_n(z) < \lambda - 1$.

Proof of Theorem 2.3 is similar to the proof of the final part of de Montessus's theorem (see [12, Section 6.2]).

Consider some applications of Theorem 2.2. The following question is related to the class of problems known as inverse problems: can certain of the poles of $\pi_{n,m}(z)$ converge to some point which is not a singular point of $a(z)$? We answer the question for the case $m = \lambda - 1$.

Proposition 8.1. *If* $\lim_{n\to\infty} a_n = a$, *where* a_n *is a pole of* $\pi_{n,\lambda-1}(z)$, *then* a *is a pole of* $a(z)$.

Proof. Suppose that a is different from the poles $z_1, \dots, z_\ell$ of the function $a(z)$. By Theorem 2.2, since a is a limit point of the set of poles for an arbitrary subsequence of $\pi_{n,\lambda-1}(z)$, a is a zero of $\omega(z,\tau)$ for all $\tau \in \mathbb{F}$.

Fix $\tau = (\tau_1, \dots, \tau_v) \in \mathbb{F}$ and consider the sequence $\tau^j = \rho^{-j}(\tau_1 z_1^j, \dots, \tau_v z_v^j) \in \mathbb{F}$, $j = 0, 1, \dots$. Denote $\omega_j(z) = \omega(z, \tau^j)$. Then it is easily seen that

$$\rho^{j+1}\omega_{j+1}(z) = \rho^j z\omega_j(z) - S_j(\tau)\Delta(z), \quad j \geqslant 0.$$

Hence $S_j(\tau)\Delta(a) = 0$. But zeros of $\omega(z,\tau)$ are different from zeros $z_1, \dots, z_v$ of $\Delta(z)$. Consequently $\Delta(a) \neq 0$ and $S_j(\tau) = 0$ for $j \geqslant 0$. However, these equalities contradict Proposition 6.1. Hence a coincides with one of the points $z_1, \dots, z_\ell$. $\square$

The proposition is a special case of Suetin's theorem [30]. Some important works in this direction are [31,32].

For another applications of Theorem 2.2 we refer to [10].

9. The geometry of the set of additional limit points

Thus the set of the limit points of poles of the sequence $\{\pi_n(z)\}$ as $n \to \infty$ consists of the poles of $a(z)$ (the multiplicity of the poles $z_1, \dots, z_v$ is less by 1), possibly the points $z = 0$ and ∞, and the set $\mathcal{N}_\mathbb{F}$ of the zeros of polynomials from the family

$$\omega(z,\tau) = \sum_{j=1}^{v} C_j \Delta_j(z)\tau_j, \quad \tau = (\tau_1, \dots, \tau_v) \in \mathbb{F}$$

(see Proposition 2.1).

The last set $\mathcal{N}_\mathbb{F}$ is called the set of *additional limit points*.

In the section we will study the geometry of this set. First we consider the case when $\mathcal{N}_\mathbb{F} = \emptyset$, i.e. when there exists $\lim_{n\to\infty} Q_n(z)$.

Proof of Theorem 2.4. Let $v = 1$, i.e. $|z_1| = \cdots = |z_\mu| > |z_{\mu+1}| \geqslant \cdots \geqslant |z_\ell|$ and $s_1 > s_2$. Then $\omega(z,\tau) = \text{const}$. Hence $\mathcal{N}_\mathbb{F} = \emptyset$ and the sequence of the $(\lambda - 1)$-normalized denominators $Q_n(z)$ has the limit:

$$\lim_{n\to\infty} Q_n(z) = (z - z_1)^{s_1 - 1}(z - z_2)^{s_2} \cdots (z - z_\ell)^{s_\ell}.$$

Conversely, suppose that $Q_n(z)$ is normalized somehow and there exists $\lim_{n\to\infty} Q_n(z)$. This hypothesis means that all polynomials from the family $\omega(z,\tau)$ have the same set of zeros for all $\tau\in\mathbb{F}$. But as we see in Proposition 8.1 the polynomials $\omega(z,\tau)$ have no common zeros. Hence $\omega(z,\tau)=\text{const}$ for all $\tau\in\mathbb{F}$. Let $v>1$. Consider the sequence $\omega_j(z)$ as in Proposition 8.1. By Proposition 6.1, since the leading coefficient of $\omega_j(z)$ is $\rho^{-j}S_j(\tau)$, there exists j such that $\deg\omega_j(z)=v-1$. This contradiction concludes the proof. □

Let us consider the next case $v=2$.

Proof of Theorem 2.5. Let $v=2$. If $z\in\mathcal{N}_{\mathbb{F}}$, i.e.

$$C_1\Delta_1(z)\tau_1+C_2\Delta_2(z)\tau_2=0,$$

then z lies on the Apollonius circle

$$\left|\frac{z-z_1}{z-z_2}\right|=\left|\frac{C_1}{C_2}\right|. \tag{9.1}$$

Now we determine when the converse inclusion is valid. We must consider the three cases. If $r=2$, i.e. $\mathbb{F}=\mathbb{T}^2$, then it is obvious that any point of circle (9.1) belongs to $\mathcal{N}_{\mathbb{F}}$.

Let now $r=1$. Suppose that Θ_0, Θ_1 are linearly independent over $\mathbb{Q}$ and $\Theta_2=\frac{p_0}{q_0}\Theta_0+\frac{p_1}{q_1}\Theta_1$, $p_j,q_j\in\mathbb{Z}$. Let us explicitly calculate the group $\mathbb{F}$. The matrix A has now the form

$$A=\begin{pmatrix} q_0 & 0 & p_0\\ 0 & q_1 & p_1\end{pmatrix}.$$

Since p_1,q_1 are coprime, there exist integers u_1,v_1 such that $q_1u_1+p_1v_1=1$. Let d be the least common multiple of the denominators q_0,q_1. Then the direct calculation shows that

$$S=\begin{pmatrix} -\dfrac{q_0p_1}{d}\\[2mm] \dfrac{q_0q_1}{d}\end{pmatrix}\quad\text{and}\quad \begin{pmatrix} -v_1 & u_1\\ q_1 & p_1\end{pmatrix}\begin{pmatrix} -\dfrac{q_0p_1}{d}\\[2mm] \dfrac{q_0q_1}{d}\end{pmatrix}=\begin{pmatrix}\dfrac{q_0}{d}\\ 0\end{pmatrix}.$$

This means that $\sigma=\frac{q_0}{d}$ and

$$T_2^{-1}=\begin{pmatrix} -v_1 & u_1\\ q_1 & p_1\end{pmatrix}.$$

Thus,

$$\mathbb{F}=\{(e^{-\frac{2\pi ijv_1}{\sigma}}t^{q_1},e^{\frac{2\pi iju_1}{\sigma}}t^{p_1})\mid t\in\mathbb{T},\ j=0,1,\dots,\sigma-1\}.$$

First let $q_1\neq p_1$. For definiteness we assume that $n=p_1-q_1>0$. If z_0 lies on circle (9.1), then

$$C_1\Delta_1(z_0)+C_2\Delta_2(z_0)t_0=0$$

for some $t_0 \in \mathbb{T}$. We take any value of the root $\sqrt[n]{t_0}$ as t_1. Multiplying the above equation by $t_1^{q_1}$, we obtain

$$\omega(z_0, \tau_0) = 0$$

for $\tau_0 = (t_1^{q_1}, t_1^{p_1}) \in \mathbb{F}$. Hence $z_0 \in \mathcal{N}_{\mathbb{F}}$. Thus in this case $\mathcal{N}_{\mathbb{F}}$ coincides with the Apollonius circle.

Let now $q_1 = p_1$. From the form of the unimodular matrix T_2^{-1} it follows that $q_1 = p_1 = 1$. Hence $\sigma = q_0$ and we can choose $v_1 = 0$, $u_1 = 1$. In this case

$$\mathbb{F} = \{t(1,\ e^{\frac{2\pi ij}{\sigma}}) \mid t \in \mathbb{T},\ j = 0, 1, \dots, \sigma - 1\}.$$

Since $\mathbb{F}$ cannot coincide with the diagonal of the torus $\mathbb{T}^2$, necessarily we have $\sigma = q_0 \neq 1$. Hence $\Theta_2 = \frac{p_0}{q_0} + \Theta_1$, i.e. the points z_1, z_2 are vertices of a regular σ-gon. Thus,

$$\omega(z, \tau) = t[C_1\Delta_1(z) + C_2 e^{\frac{2\pi ij}{\sigma}}\Delta_2(z)], \quad t \in \mathbb{T},\ j = 0, 1, \dots, \sigma - 1,$$

and $\mathcal{N}_{\mathbb{F}}$ coincides with the set of zeros of the polynomials

$$\omega_j(z) = C_1(z - z_2) + C_2 e^{\frac{2\pi ij}{\sigma}}(z - z_1), \quad j = 0, 1, \dots, \sigma - 1,$$

i.e. $\mathcal{N}_{\mathbb{F}}$ consists of a finite number of points lying on curve (9.1). If all $\tau \in \mathbb{F}$ are regular, then the number of these points equals σ. The singular point τ exists iff $C_1 + C_2 e^{\frac{2\pi ij}{\sigma}} = 0$ for some (unique) value of j, $j = 0, 1, \dots, \sigma - 1$. In this case the number of points in $\mathcal{N}_{\mathbb{F}}$ equals $\sigma - 1$.

It remains to consider the case $r = 0$. Then $\Theta_1 = \frac{p_1}{q_1},\ \Theta_2 = \frac{p_2}{q_2}$, $p_j, q_j \in \mathbb{Z}$, and z_1, z_2 are also vertices of a regular σ-gon, where σ is the least common multiple of the denominators q_1 and q_2. Let $\Theta_1 = \frac{n_1}{\sigma},\ \Theta_2 = \frac{n_2}{\sigma}$. Then

$$\mathbb{F} = \{(e^{\frac{2\pi in_1 j}{\sigma}}, e^{\frac{2\pi in_2 j}{\sigma}}) \mid j = 0, 1, \dots, \sigma - 1\}.$$

Therefore $\mathcal{N}_{\mathbb{F}}$ consists of σ or $\sigma - 1$ (in the singular case) points. The theorem is proved. $\square$

Let now $v > 2$. First we describe general properties of $\mathcal{N}_{\mathbb{F}}$ that hold for any group $\mathbb{F}$. Denote by $\mathcal{N}_j$ $(1 \leqslant j \leqslant v)$ the set of complex points z satisfying the inequalities

$$|C_j\Delta_j(z)| \leqslant \sum_{\substack{k=1\\k\neq j}}^{v} |C_k\Delta_k(z)|, \quad j = 1, \dots, v.$$

Put $\mathcal{N} = \bigcap_{j=1}^{v} \mathcal{N}_j$. We note that the poles $z_1, \dots, z_v$ do not belong to $\mathcal{N}$ because $C_j\Delta_j(z_j)\tau_j \neq 0$, $\Delta_k(z_j) = 0$, $k \neq j$, $j, k = 1, \dots, v$.

Proposition 9.1. *For $v > 2$ the set $\mathcal{N}_{\mathbb{F}}$ is a nonempty closed subset of $\mathcal{N}$.*

The proof is straightforward.

In particular, for $v = 2$ the set $\mathcal{N}$ is the Apollonius circle. For $v > 2$ the set $\mathcal{N}$ already does not degenerate into a curve. The boundary of $\mathcal{N}$ is the curve $L = \bigcup_{j=1}^{v} L_j$, where L_j is defined by the equation

$$|C_j \Delta_j(z)| = \sum_{\substack{k=1 \\ k \neq j}}^{v} |C_k \Delta_k(z)|, \quad j = 1, \dots, v.$$

It is not difficult to verify the following properties of L_j:

(1) $L_i \cap L_j = \emptyset$, $i \neq j$;
(2) L_j is symmetric with respect to the circle $|z| = \rho$;
(3) L_j is unbounded iff

$$|C_j| = \sum_{\substack{k=1 \\ k \neq j}}^{v} |C_k|.$$

In order to prove Theorem 2.6, we need a criterion for the existence of a root of a polynomial $\alpha(\tau) = \alpha_0 + \alpha_1 \tau_1 + \cdots + \alpha_n \tau_n$ belonging to the torus $\mathbb{T}^n$. We will use the following simple homotopic considerations.

Let $\alpha(\tau_1, \dots, \tau_n)$ be continuous and nonvanishing on the torus $\mathbb{T}^n$ function. Fix values of $\tau_1, \dots, \tau_{n-1}$: $\tau_1 = \tau_1^0, \dots, \tau_{n-1} = \tau_{n-1}^0$. Define the partial Cauchy index with respect to the variable τ_n:

$$\kappa_n = \frac{1}{2\pi} [\arg \alpha(\tau_1^0, \dots, \tau_{n-1}^0, e^{i\Theta})]_{\Theta=0}^{2\pi}.$$

Here $[\arg]_{\Theta=0}^{2\pi}$ is the increment of arg when Θ changes from $\Theta = 0$ to 2π. Since the torus $\mathbb{T}^{n-1}$ is a compact connected set, the index κ_n does not depend on a way of fixing of $\tau_1, \dots, \tau_{n-1}$. In particular, if $\alpha(\tau_1, \dots, \tau_n)$ is a polynomial, then $\alpha(\tau_1^0, \dots, \tau_{n-1}^0, \tau_n)$ has the same numbers of zeros into the unit disk $|\tau| < 1$ for any way of fixing of $\tau_1, \dots, \tau_{n-1}$.

Lemma 9.1. *A polynomial $\alpha(\tau) = \alpha_0 + \alpha_1 \tau_1 + \cdots + \alpha_n \tau_n$, $\alpha_j \neq 0$ for $j = 1, \dots, n$, has a root belonging to the torus $\mathbb{T}^n$ if and only if the inequalities*

$$|\alpha_j| \leqslant \sum_{\substack{k=0 \\ k \neq j}}^{n} |\alpha_k|, \quad j = 0, \dots, n \tag{9.2}$$

are valid.

Proof. The proof of necessity is trivial. Let us prove sufficiency.

Obviously, for $n = 1$ inequalities (9.2) provide the existence of a root of $\alpha(\tau)$ in $\mathbb{T}^n$. Let $n > 1$ and $\alpha_j = |\alpha_j| e^{i\varphi_j}$, $j = 0, 1, \dots, n$. First we suppose that for the coefficients of the polynomial $\alpha(\tau)$ the equality

$$|\alpha_j| = \sum_{\substack{k=0\\k\neq j}}^{n} |\alpha_k|$$

holds for some j, $0\leqslant j\leqslant n$. Put

$$\tau_j^0 = -e^{-i(\varphi_j-\varphi_0)}, \quad \tau_k^0 = e^{-i(\varphi_j-\varphi_0)}, \quad k\neq j.$$

Then $\alpha(\tau_1^0,\dots,\tau_n^0)=0$. Hence in this case there exists a root $\tau^0=(\tau_1^0,\dots,\tau_n^0)$ in $\mathbb{T}^n$. Suppose now that the strict inequalities

$$|\alpha_j| < \sum_{\substack{k=0\\k\neq j}}^{n} |\alpha_k|, \quad j=0,1,\dots,n \tag{9.3}$$

are valid.

The existence of a root we prove by induction on $n\geqslant 2$. Let $n=2$. At first we fix $\tau_1^0 = e^{-i(\varphi_1-\varphi_0)}$. Then $\alpha(\tau_1^0,\tau_2)$ has a unique root $\tau_2^0 = -e^{i\varphi_0}\frac{|\alpha_0|+|\alpha_1|}{\alpha_2}$, and, by inequalities (9.3), we have $|\tau_2^0|>1$. Hence $\alpha(\tau_1^0,\tau_2)$ has no roots inside the unit circle $|\tau_2|=1$. Now we fix the other value of $\tau_1^1=-e^{-i(\varphi_1-\varphi_0)}$. In this case $\alpha(\tau_1^1,\tau_2)$ has a unique root $\tau_2^1 = -e^{i\varphi_0}\frac{|\alpha_0|-|\alpha_1|}{\alpha_2}$, and, in virtue of inequalities (9.3), we obtain $|\tau_2^1|<1$. Therefore $\alpha(\tau_1^1,\tau_2)$ has one root inside the circle $|\tau_2|=1$. If we suppose that $\alpha(\tau_1,\tau_2)\neq 0$ on the torus $\mathbb{T}^2$, then $\alpha(\tau_1,\tau_2)$ has different numbers of roots inside $|\tau_2|=1$ for different ways of fixing of τ_1. But this is impossible and hence $\alpha(\tau_1,\tau_2)$ has a root belonging to $\mathbb{T}^2$.

Suppose our statement holds if the number of variables of $\alpha(\tau)$ is $n-1$. Prove its validity for polynomials in n variables. At first we fix $\tau_1,\dots,\tau_n$ as follows: $\tau_1^0 = e^{-i(\varphi_1-\varphi_0)},\dots,\tau_{n-1}^0 = e^{-i(\varphi_{n-1}-\varphi_0)}$. Then the polynomial

$$\alpha(\tau_1^0,\dots,\tau_{n-1}^0,\tau_n) = (|\alpha_0|+\cdots+|\alpha_{n-1}|)e^{i\varphi_0}+\alpha_n\tau_n$$

has a unique root

$$\tau_n^0 = -e^{i\varphi_0}\frac{|\alpha_0|+\cdots+|\alpha_{n-1}|}{\alpha_n},$$

lying, in virtue of the inequality $|\alpha_n|<|\alpha_0|+\cdots+|\alpha_{n-1}|$, outside the disk $|\tau_n|\leqslant 1$. If for the coefficients $\alpha_0,\dots,\alpha_{n-1}$ the inequalities

$$|\alpha_j| \leqslant \sum_{\substack{k=0\\k\neq j}}^{n-1} |\alpha_k|, \quad j=0,\dots,n-1, \tag{9.4}$$

are valid, then, by the induction hypothesis, there is a point $(\tau_1^1,\dots,\tau_{n-1}^1)\in\mathbb{T}^{n-1}$ such that $\alpha_0+\alpha_1\tau_1^1+\cdots+\alpha_{n-1}\tau_{n-1}^1=0$. Hence the polynomial

$$\alpha(\tau_1^1,\dots,\tau_{n-1}^1,\tau_n) = \alpha_n\tau_n$$

has the root $\tau_n^1 = 0$ lying inside $|\tau_n| = 1$. This is possible only if $\alpha(\tau_1, \dots, \tau_n)$ has a root in $\mathbb{T}^n$. Hence if conditions (9.4) are fulfilled, the statement of the lemma is proved.

Suppose now that these conditions are not fulfilled, i.e. there is some j, $0 \leqslant j \leqslant n-1$, such that

$$|\alpha_j| > \sum_{\substack{k=0 \\ k \neq j}}^{n-1} |\alpha_k|.$$

Fix now values of $\tau_1, \dots, \tau_{n-1}$ as follows: $\tau_j^1 = -e^{-i(\varphi_j - \varphi_0)}$, $\tau_k^1 = e^{-i(\varphi_k - \varphi_0)}$, $k = 1, \dots, n-1$, $k \neq j$.

Then the polynomial

$$\alpha(\tau_1^1, \dots, \tau_{n-1}^1, \tau_n) = e^{i\varphi_0} \left(\sum_{\substack{k=0 \\ k \neq j}}^{n-1} |\alpha_k| - |\alpha_j| \right) + \alpha_n \tau_n$$

has a unique root τ_n^1 for which, in virtue of our supposition, holds

$$|\tau_n^1| = \frac{|\alpha_j| - \sum\limits_{\substack{k=0 \\ k \neq j}}^{n-1} |\alpha_k|}{|\alpha_n|}.$$

Taking into account the inequality

$$|\alpha_j| < \sum_{\substack{k=0 \\ k \neq j}}^{n} |\alpha_k|$$

from the system of inequalities (9.2), we get $|\tau_n^1| < 1$. Again this is possible iff $\alpha(\tau_1, \dots, \tau_n)$ has a root in $\mathbb{T}^n$. The lemma is completely proved. □

Theorem 2.6 is a direct consequence of the lemma.

Proof of Theorem 2.6. It remains to prove only the inclusion $\mathcal{N} \subseteq \mathcal{N}_{\mathbb{F}}$. If $z_0 \in \mathcal{N}$, i.e.,

$$|C_j \Delta_j(z_0)| \leqslant \sum_{\substack{k=1 \\ k \neq j}}^{v} |C_k \Delta_k(z_0)|, \quad j = 1, \dots, v,$$

then, by Lemma 9.1, the polynomial $\omega(z_0, \tau)$ has a root $\tau^0 = (\tau_1^0, \dots, \tau_n^0) \in \mathbb{T}^v = \mathbb{F}$. Hence $z_0 \in \mathcal{N}_{\mathbb{F}}$. □

Lemma 9.1 allows to find $\mathcal{N}_{\mathbb{F}}$ for one more case.

Theorem 9.1. *Suppose $\Theta_0 = 1, \Theta_1, \dots, \Theta_r$ $(1 \leqslant r < v)$ are linearly independent over $\mathbb{Q}$ and Θ_k $(r+1 \leqslant k \leqslant v)$ are rational numbers. Since Θ_k is defined up to an integer summand, we can uniquely represent it as follows*:

$$\Theta_k = \frac{n_k}{\sigma}, \quad k = r+1, \dots, v,$$

where $0 \leqslant n_k \leqslant \sigma - 1$. *By* $\mathcal{N}^n$ *we denote the set of complex points satisfying the inequalities*:

$$|C_j \Delta_j(z)| \leqslant \sum_{\substack{k=0\\k \neq j}}^{r} |C_k \Delta_k(z)|, \quad j = 0, 1, \ldots, r.$$

Here for uniformity we put

$$C_0 \Delta_0(z) = |\rho|^{-n} \sum_{k=r+1}^{v} C_k \Delta_k(z) z_k^n, \quad n = 0, 1, \ldots, \sigma - 1.$$

Then

$$\mathcal{N}_{\mathbb{F}} = \bigcup_{n=0}^{\sigma-1} \mathcal{N}^n.$$

Proof. The group $\mathbb{F}$ in this case can be easily found by definition:

$$\begin{aligned}\mathbb{F} &= \overline{\{(e^{2\pi i n \Theta_1}, \ldots, e^{2\pi i n \Theta_v})\}}_{n \geqslant 0}\\ &= \{(\tau_1, \ldots, \tau_r, \xi_n^{n_{r+1}}, \ldots, \xi_n^{n_v}) \mid (\tau_1, \ldots, \tau_r) \in \mathbb{T}^r,\ n = 0, 1, \ldots, \sigma - 1\},\end{aligned}$$

where $\xi_n = e^{\frac{2\pi i n}{\sigma}}$. Since $(\xi_n^{n_{r+1}}, \ldots, \xi_n^{n_v}) = \rho^{-n}(z_{r+1}^n, \ldots, z_v^n)$, we have

$$\omega(z, \tau) = \sum_{k=1}^{r} C_k \Delta_k(z) \tau_k + \rho^{-n} \sum_{k=r+1}^{v} C_k \Delta_k(z) z_k^n,$$

$(\tau_1, \ldots, \tau_r) \in \mathbb{T}^r$, $n = 0, 1, \ldots, \sigma - 1$.

As above it remains to apply Lemma 9.1. □

We conclude the section with the case of a regular arrangement of the poles $z_1, \ldots, z_v$.

Proof of Theorem 2.7. Suppose that $\mathcal{N}_{\mathbb{F}}$ consists of a finite number of points. Fix any point $(\tau_1, \ldots, \tau_v) \in \mathbb{F}$ and consider the sequence of polynomials $\omega_j(z)$, $j \geqslant 0$, as in Proposition 8.1. From the recurrence relation $\rho^{j+1} \omega_{j+1}(z) = \rho^j z \omega_j(z) - S_j(\tau) \Delta(z)$ we obtain

$$\rho^{j+k} \omega_{j+k}(z) = \rho^j z^k \omega_j(z) - \sum_{i=0}^{k-1} z^{k-i-1} S_{j+i}(\tau) \Delta(z). \tag{9.5}$$

Since $\mathcal{N}_{\mathbb{F}}$ is finite, in the sequence $\{\omega_j(z)\}_{j=0}^{\infty}$ there are polynomials that coincides up to constant factor. Let $\omega_{j+\sigma}(z) = c\omega_j(z)$. It follows from Eq. (9.5) that

$$\rho^j (c\rho^\sigma - z^\sigma) \omega_j(z) = \sum_{i=0}^{\sigma-1} z^{\sigma-i-1} S_{j+i}(\tau) \Delta(z).$$

Since $\omega_j(z)$ and $\Delta(z)$ are coprime, $\Delta(z)$ is a divisor of $c\rho^\sigma - z^\sigma$. Hence $|c| = 1$ and all roots $z_1, \ldots, z_v$ of the polynomial $\Delta(z)$ lie in vertices of a regular σ-gon.

Conversely, suppose the poles $z_1, \dots, z_v$ are vertices of a regular σ-gon. Let us take the pole $z_1 = \rho e^{2\pi i \Theta_1}$ as the original vertex. Then the arguments of the vertices are $\Theta_1, \Theta_1 + \frac{1}{\sigma}, \dots, \Theta_1 + \frac{\sigma-1}{\sigma}$ and

$$\Theta_k = \Theta_1 + \frac{n_k}{\sigma}, \quad 2 \leqslant k \leqslant v \leqslant \sigma, \ 1 \leqslant n_k \leqslant \sigma - 1.$$

As above the group $\mathbb{F}$ can be easily found by the definition. Let $\mathbb{F}_1$ be the closure of the cyclic group generated by $e^{2\pi i \Theta_1}$. It is clear that $\mathbb{F}_1$ is a finite cyclic group if Θ_1 is a rational number, and $\mathbb{F}_1 = \mathbb{T}$ if Θ_1 is an irrational one. Since $e^{\frac{2\pi i n_k}{\sigma}} = \frac{z_k}{z_1}$, we have

$$\mathbb{F} = \{\xi z_1^{-j}(z_1^j, \dots, z_v^j) \mid \xi \in \mathbb{F}_1, \ j = 0, 1, \dots, \sigma - 1, \}$$

and $\omega(z, \tau) = \xi z_1^{-j} \omega_j(z)$, where $\omega_j(z) = \sum_{k=1}^{v} C_k \Delta_k(z) z_k^j, \ j = 0, \dots, \sigma - 1$. Hence $\mathcal{N}_{\mathbb{F}}$ is the union of the zeros of the polynomials $\omega_j(z), \ j = 0, \dots, \sigma - 1$, i.e. a finite set.

The sequence Λ_τ now coincides with the sequence Λ_j consisting of the positive integers n such that $n \equiv j \pmod{\sigma}$. The polynomials $\omega_n(z)$ can be defined for all $n \geqslant 0$. Let $\alpha_n = \sum_{k=1}^{v} C_k z_k^n$ be the (formal) leading coefficient of the polynomial $\omega_n(z)$. Then the following recurrence formula of type (4.3):

$$\omega_{n+1}(z) = z\omega_n(z) - \alpha_n \Delta(z), \quad n \geqslant 0,$$

is evident. □

It is not difficult to show that if $r = 1$, then the set $\mathcal{N}_{\mathbb{F}}$ consists of a finite number of points (for the case when the connected component of the identity is a diagonal subgroup of $\mathbb{F}$) or a finite numbers of algebraic curves. However an explicit description of $\mathcal{N}_{\mathbb{F}}$ for any group $\mathbb{F}$ is not yet obtained.

Acknowledgments

I would like to thank an anonymous referee who mentioned very useful works of A. Sidi, X. Liu and E.B. Saff on intermediate rows. I am very grateful to the editor of the manuscript who sent copies of these works. Also I wish to express my gratitude to the referee for pointing out a number of typos and mistakes in the first version of the paper.

References

[1] V.M. Adukov, Factorization of triangular 2 × 2 matrix functions, Manuscript 5930-82, deposited at VINITI, 1982 (in Russian).
[2] V.M. Adukov, The structure of the kernel and inversion of block Toeplitz matrices, Manuscript 9030-B85, deposited at VINITI, 1985 (in Russian).
[3] V.M. Adukov, On indices and essential polynomials, Izv. Vuzov. Mat. (6) (1986) 47–49 (in Russian); English transl. in Soviet Math. (IZ. VUZ) 30 (1986).
[4] V.M. Adukov, The structure of the kernel and inversion of Toeplitz and Hankel matrices, Izv. Vuzov. Mat. (7) (1986) 3–8 (in Russian); English transl. in Soviet Math. (IZ. VUZ) 30 (1986).

[5] V.M. Adukov, On Wiener–Hopf factorization of meromorphic matrix functions, Integral Equations Operator Theory 14 (1991) 767–774.
[6] V.M. Adukov, Wiener–Hopf factorization of meromorphic matrix functions, Algebra i Anal 4 (1) (1992) 54–74 (in Russian); English transl. St. Petersburg Math. J. 4 (1) (1993) 51–69.
[7] V.M. Adukov, Generalized inversion of block Toeplitz matrices, Linear Algebra Appl. 274 (1998) 85–124.
[8] V.M. Adukov, On factorization of analytic matrix functions, Teoret. Mat. Fiz. 118 (3) (1999) 324–336 (in Russian); English transl. Theoret. and Math. Phys. 118 (3) (1999) 255–263.
[9] V.M. Adukov, The essential polynomial approach to convergence of matrix Padé approximants, Contemp. Math. 280 (2001) 71–87.
[10] V.M. Adukov, On existence of convergent subsequences of a row of the Padé table for meromorphic functions, Proceedings of the Chelyabinsk Scientific Center No. 3, 2002, pp. 1–5 (in Russian).
[11] G.A. Baker Jr., P.R. Graves-Morris, Convergence of rows of the Padé table, J. Math. Anal. Appl. 57 (1977) 323–339.
[12] G.A. Baker Jr., P.R. Graves-Morris, Padé Approximants, Addison–Wesley, Reading, MA, 1981.
[13] N. Bourbaki, Éléments de mathématique, Livre III, Topologie générale, Hermann & Cie, Paris, 1940–1949.
[14] V.I. Buslaev, A.A. Gončar, S.P. Suetin, On convergence of subsequences of the *m*th row of the Padé table, Mat. Sb. 120 (4) (1983) 540–545 (in Russian).
[15] R.L. Ellis, D.C. Lay, Factorization of finite rank Hankel and Toeplitz matrices, Linear Algebra Appl. 173 (1992) 19–38.
[16] A.O. Gelfond, Calculus of Finite Differences, GITTL, Moscow, Leningrad, 1952 (in Russian).
[17] I.C. Gohberg, N.Ya. Krupnik, Einführung in die Theorie der Eindimensionalen Singulären Integraloperatoren, Birkhäuser, Boston, 1979.
[18] A.A. Gončar, On convergence of the Padé approximants for some classes of meromorphic functions, Mat. Sb. 97 (4) (1975) 607–629 (in Russian).
[19] A.A. Gončar, Poles of rows of the Padé table and a meromorphic extension of functions, Mat. Sb. 115 (4) (1981) 590–613 (in Russian).
[20] J. Hadamard, Essai sur l'étude des fonctions données par leur développement de Taylor, J. Math. Pures Appl. 4 (8) (1892) 101–186.
[21] A.G. Kurosh, The Theory of Groups, Moscow, Nauka, 1967 (in Russian).
[22] J. Lin, Extension of Wilson's theorem, Comput. Math. 1 (1987) 43–48 (in Chinese).
[23] X. Liu, Generalizations of Montessus's theorem on the row convergence of rational interpolations, Methods Appl. Anal. 2 (1995) 442–465.
[24] X. Liu, E.B. Saff, Intermediate rows of the Walsh array of best rational approximants to meromorphic functions, Methods Appl. Anal. 2 (1995) 269–284.
[25] R. de Montessus de Ballore, Sur les fractions continues algébriques, Bull. Soc. Math. de France 30 (1902) 28–36.
[26] S.A. Morris, Pontryagin Duality and the Structure of Locally Compact Abelian Groups, in: London Mathematical Society Lecture Note Series, No. 29, Cambridge University Press, Cambridge, 1977.
[27] E.A. Rakhmanov, On convergence of the diagonal Padé approximants, Mat. Sb. 104 (2) (1977) 271–291 (in Russian).
[28] E.B. Saff, On the row convergence of the Walsh array for meromorphic functions, Trans. Amer. Math. Soc. 146 (1969) 241–257.
[29] A. Sidi, Quantitative and constructive aspects of the generalized Koenig's and de Montessus's theorems for Padé approximants, J. Comput. Appl. Math. 29 (1990) 257–291.
[30] S.P. Suetin, On poles of the *m*th row of the Padé table, Mat. Sb. 120 (4) (1983) 500–504 (in Russian).
[31] S.P. Suetin, On an inverse problem for the *m*th row of the Padé table, Mat. Sb. 124 (2) (1984) 238–250 (in Russian).
[32] V.V. Vavilov, G. Lopes, V.A. Prokhorov, On an inverse problem for rows of the Padé table, Mat. Sb. 110 (1) (1979) 117–127 (in Russian).
[33] B.L. van der Waerden, Algebra, Teil II, 5 aufl., Springer, Berlin, Heidelberg, New York, 1967.